Fourier Series and Sturm-Liouville Eigenvalue Problems

Y. K. Goh

2009
Outline

- Functions
- Fourier Series Representation
- Half-range Expansion
- Convergence of Fourier Series
- Parseval’s Theorem and Mean Square Error
- Complex Form of Fourier Series
- Inner Products
- Orthogonal Functions
- Self-adjoint Operators
- Sturm-Liouville Eigenvalue Problems
Definition (Periodic Function)

A $2L$-periodic function $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is a function such that there exists a constant $L > 0$ such that

$$f(x) = f(x + 2L), \quad \forall x \in \mathbb{R}. \quad (1)$$

Here $2L$ is called the fundamental period or just period. For example, $f(x) = \sin(x)$ is a $2\pi$-periodic function with period $2\pi$, since $\sin(x + 2\pi) = \sin(x)$. 

Y. K. Goh

Fourier Series and Sturm-Liouville Eigenvalue Problems
Even and Odd Functions

Definition (Even and Odd Functions)

- A function \( f \) is **even** if and only if \( f(-x) = f(x), \forall x \).
- A function \( f \) is **odd** if and only if \( f(-x) = -f(x), \forall x \).

For example

- \( \sin x \) is an odd function since \( \sin(-x) = -\sin x \).
- \( \cos x \) is an even function since \( \cos(-x) = \cos x \).

Note that even function is symmetric about the \( y \)-axis. On the other hand, odd function is symmetric about the origin.
Examples of Even and Odd Functions

- Even function
- Odd function
- Odd function
- Even function
Definition (Piecewise Continuous Functions)

A function $f$ is said to be piecewise continuous on the interval $[a, b]$ if

1. $f(a^+)$ and $f(b^-)$ exist, and
2. $f$ is defined and continuous on $(a, b)$ except at a finite number of points in $(a, b)$ where the left and right limits exits
Definition (Piecewise Smooth Functions)

A function \( f \), defined on the interval \([a, b]\), is said to be piecewise smooth if \( f \) and \( f' \) are piecewise continuous on \([a, b]\).

Thus \( f \) is piecewise smooth if

1. \( f \) is piecewise continuous on \([a, b]\),

2. \( f' \) exists and is continuous in \((a, b)\) except possibly at finitely many points \( c \) where the one-sided limits \( \lim_{x \to c^-} f'(x) \) and \( \lim_{x \to c^+} f'(x) \) exist. Furthermore, \( \lim_{x \to a^+} f'(x) \) and \( \lim_{x \to b^-} f'(x) \) exist.

Y. K. Goh

Fourier Series and Sturm-Liouville Eigenvalue Problems
Examples of Piecewise Functions

Figure: A piecewise smooth function.

Figure: Another piecewise smooth function.
Some useful integrations

- If $f$ is even. Then, for any $a \in \mathbb{R}$

$$\int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx.$$  

- If $f$ is odd. Then, for any $a \in \mathbb{R}$

$$\int_{-a}^{a} f(x) \, dx = 0.$$  

- If $f$ is piecewise continuous and $2L$-periodic. Then, for any $a \in \mathbb{R}$

$$\int_{0}^{2L} f(x) \, dx = \int_{a}^{a+2L} f(x) \, dx.$$
Orthogonal Functions

Definition (Orthogonal Functions)
Two functions $f$ and $g$ are said to be orthogonal in the interval $[a, b]$ if

$$\int_{a}^{b} f(x)g(x) \, dx = 0.$$  \hspace{1cm} (2)

We will come back to orthogonal functions again later.
Orthogonal Properties of Trigonometric Functions

The orthogonal properties of sine and cosine functions are summarised as follow:

\[\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0, \quad (3)\]

\[\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \pi \delta_{mn}, \quad (4)\]

\[\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \pi \delta_{mn} \quad (5)\]

where \(\delta_{mn}\) is the Kronecker’s delta.
Kronecker’s Delta

Definition (Kronecker’s Delta)

\[ \delta_{mn} = \begin{cases} 
1, & m = n \\
0, & m \neq n.
\end{cases} \quad (6) \]
Theorem (Fourier Series Representation)

Suppose $f$ is a $2L$-periodic piecewise smooth function, then Fourier series of $f$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x)$$  \hspace{2cm} (7)

and the Fourier series converges to $f(x)$ if $f$ is continuous at $x$ and to $\frac{1}{2}[f(x+) + f(x-)]$ otherwise.

Here $\omega = \frac{2\pi}{2L}$ is called the fundamental frequency, while the amplitudes $a_0$, $a_n$, and $b_n$ are called Fourier coefficients of $f$ and they are given by the Euler formula.
Euler Formula

Definition (Euler Formula)

The Fourier coefficients for a $2L$-periodic function $f$ are given by

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos n\omega x \, dx = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin n\omega x \, dx = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.$$

for $n = 1, 2, \ldots.$
Collorary

- If $f$ is even and $2L$-periodic, then the Fourier series representation is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega x.$$ 

- If $f$ is odd and $2L$-periodic, then the Fourier series representation is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\omega x.$$ 

Here, $a_0$, $a_n$, and $b_n$ are given by the Euler formula.
Examples of Fourier Series

(Odd function, digital impulses) Find the Fourier representation of the periodic function \( f(x) \) with period \( 2\pi \), where

\[
f(x) = \begin{cases} 
-1, & -\pi < x < 0, \\
1, & 0 < x < \pi.
\end{cases}
\]

[Answer:]

\[
f(x) = \sum_{n \text{ odd}}^{\infty} \frac{4 \sin nx}{n\pi} = \sum_{k=1}^{\infty} \frac{4 \sin(2k - 1)x}{(2k - 1)\pi}.
\]
Graphs for Digital Pulse Train and its Fourier Series

Y. K. Goh

Fourier Series and Sturm-Liouville Eigenvalue Problems
(Even function) Find the Fourier series for \( f(x) = |x| \) if \(-1 < x < 1\) and \( f(x + 2) = f(x) \).

[Answer:]

\[
f(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4 \cos(2n - 1)x}{(2n - 1)^2 \pi^2}.
\]
Graphs for $f(x) = |x|$, $f(x + 2) = f(x)$
Find the Fourier series of the 2-periodic function
\[ f(x) = x^3 + \pi \text{ if } -1 < x < 1. \]

[Answer:]
\[ f(x) = \pi + \sum_{n=1}^{\infty} (-1)^n \frac{12 - 2n^2\pi^2}{n^3\pi^3} \sin n\pi x. \]
Graphs for $f(x) = x^3 + \pi$, $f(x + 2) = f(x)$
Consider a function $f$ that is only defined in the interval $[0, p)$. We could always extension the function outside the range to produce a new function. Of course, we have infinite many ways to extend the function, but here we will focus only on three specific extensions.

**Definition (Full-range Periodic Extension)**

The full-range periodic extension $g$ of a function $f$ defined in $[0, p)$ is a $p$-periodic function given by

$$
\begin{align*}
g(x) &= f(x) & \text{if } 0 \leq x < p, \\
g(x) &= g(x + p) & \text{if } x < 0, \\
g(x) &= g(x - p) & \text{if } x \geq p.
\end{align*}
$$
Definition (Half-range Even Periodic Extension)

The half-range even periodic extension \( f_e \) of a function \( f \) defined in \([0, p)\) is a \(2p\)-periodic even function given by

\[
f_e(x) = \begin{cases} 
  f(x) & 0 \leq x < p, \\
  f(-x) & -p \leq x < 0.
\end{cases}
\]
Definition (Half-range Odd Periodic Extension)

The half-range odd periodic extension $f_o$ of a function $f$ defined in $[0, p)$ is a $2p$-periodic odd function given by

$$f_e(x) = \begin{cases} 
  f(x) & 0 \leq x < p, \\
  -f(-x) & -p \leq x < 0.
\end{cases}$$
Examples Periodic Extensions

- **f(x) defined in [0, 1]**
  - Half-range even extension
  - Full-range periodic extension

- **Half-range odd extension**

Y. K. Goh

Fourier Series and Sturm-Liouville Eigenvalue Problems
Full-range Fourier Series for $f$ defined on $[0, p)$

**Theorem**

If $f(x)$ is a piecewise smooth function defined on an interval $[0, p)$, then $f$ has a full-range Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x), \quad 0 \leq x < p, \quad (8)$$

where $\omega = \frac{2\pi}{p}$ and the Fourier coefficients

$$a_0 = \frac{1}{p/2} \int_{0}^{p} f(x) \, dx, \quad a_n = \frac{1}{p/2} \int_{0}^{p} f(x) \cos n\omega x \, dx, \quad \text{and}$$

$$b_n = \frac{1}{p/2} \int_{0}^{p} f(x) \sin n\omega x \, dx.$$
Fourier Cosine Series for $f$ defined on $[0, p)$

**Theorem**

If $f(x)$ is a piecewise smooth function defined on an interval $[0, p)$, then $f$ has a half-range Fourier cosine series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega x, \quad 0 \leq x < p,$$

where $\omega = \frac{\pi}{p}$ and the Fourier coefficients $a_0 = \frac{2}{p} \int_{0}^{p} f(x) \, dx$, and $a_n = \frac{2}{p} \int_{0}^{p} f(x) \cos n\omega x \, dx$. 

Y. K. Goh

Fourier Series and Sturm-Liouville Eigenvalue Problems
Fourier Since Series for $f$ defined on $[0, p)$

**Theorem**

If $f(x)$ is a piecewise smooth function defined on an interval $[0, p)$, then $f$ has a half-range Fourier sine series expansion

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\omega x, \quad 0 \leq x < p,$$

(10)

where $\omega = \frac{\pi}{p}$ and the Fourier coefficients

$$b_n = \frac{2}{p} \int_{0}^{p} f(x) \sin n\omega x \, dx.$$
Consider a signal $f(t) = t$ measured from an experiment over the duration given by $0 \leq t < 4$.

- Sketch the full-range periodic extension of $f(t)$. Find the corresponding Fourier expansion of $f(t)$.
- Sketch the half-range even extension of $f(t)$ and find the corresponding Fourier cosine expansion of $f(t)$.
- Sketch the half-range odd extension of $f(t)$ and find the corresponding Fourier sine expansion of $f(t)$. 
Convergence of Fourier Series

In the Fourier Series Representation Theorem, we were saying that for every $2L$-periodic piecewise smooth function $f$, we could construct a partial sum

$$s_N(x) = \frac{a}{2} + \sum_{n=1}^{N} (a_n \cos n\omega x + b_n \sin n\omega x).$$

And, when $N \to \infty$, the partial sum $s_N(x)$ converges to

- $f(x)$, if $f(x)$ is continuous for all $x$;
- $\frac{1}{2}[f(x+) + f(x-)]$, at the discontinuous points, or jumps.
\textbf{Figure:} $s_N(x)$ converges to $f(x)$, except at the jumps.

\textbf{Figure:} $s_N(x)$ converges uniformly on the interval $[-1, 1]$.
Definition (Pointwise Convergence)

A sequence of functions \(\{s_n\}\) is said to converge pointwise to the function \(f\) on the set \(E\), if the sequence of numbers \(\{s_n(x)\}\) converges to the number \(f(x)\), for each \(x\) in \(E\).

Or,

if \(\forall x \in E, s_n(x) \to f(x)\), then \(\{s_n\}\) converge pointwise to \(f\).
Uniform Convergence

Definition (Uniform Convergence)

We say that $s_n$ converges to $f$ uniformly on a set $E$, and we write $s_n \to f$ uniformly on $E$ if, given $\epsilon > 0$, we can find a positive integer $N$ such that for all $n \geq N$

$$|s_n(x) - f(x)| < \epsilon, \forall x \in E.$$ 

Definition (Uniform Convergence Series)

A series $s(x) = \sum_{k=0}^{\infty} u_k(x)$ is said to converge uniformly to $f(x)$ on a set $E$ if the sequence of partial sums $s_n(x) = \sum_{k=0}^{n} u_k(x)$ converges uniformly to $f(x)$. 

Y. K. Goh

Fourier Series and Sturm-Liouville Eigenvalue Problems
Note that if a sequence of partial sums $s_n$ converges uniformly to $f$, then $s_n$ is also pointwise convergence. However, the converse is not always true.

In order to determine if $s_n$ is uniformly convergence, we use the Weierstraß $M$-test.
Theorem (Weierstraß $\mathcal{M}$-Test)

Let $\{u_k\}_{k=0}^{\infty}$ be a sequence of real- or complex-valued functions on $E$. If there exists a sequence $\{M_k\}_{k=0}^{\infty}$ of nonnegative real numbers such that the following two conditions hold:

- $|u_k(x)| \leq M_k$, $\forall x \in E$, and
- $\sum_{k=0}^{\infty} M_k < \infty$.

Then $\sum_{k=0}^{\infty} u_k(x)$ converges uniformly on $E$. 

Y. K. Goh

Fourier Series and Sturm-Liouville Eigenvalue Problems
Gibbs’ Phenomena

Here is an example of non-uniform convergence. The peaks remain same height but the width of the peaks changes.
Mean Square Error

Since $s_N$ converges to $f$ only when $N \to \infty$, for most practical purposes, we need $N$ to be large but finite. Thus, we are approximating $f$ with $s_N$, and it is important for us to keep track of the error of the approximation.

**Definition (Mean Square Error)**

The mean square error of the partial sum $s_N$ relative to $f$ is

$$E_N = \frac{1}{2L} \int_{-L}^{L} [f(x) - s_N(x)]^2 \, dx$$

$$= \frac{1}{2L} \int_{-L}^{L} [f(x)]^2 \, dx - \frac{1}{4} a_0^2 - \frac{1}{2} \sum_{n=1}^{N} (a_n^2 + b_n^2).$$
Mean Square Approximation

Theorem (Mean Square Approximation)

Suppose that $f$ is square integrable, i.e. $\int_{-L}^{L} |f(x)|^2 \, dx$ on $[-L, L]$. Then $s_N$, the $N$th partial sum of the Fourier series of $f$, approximates $f$ in the mean square sense with an error $E_N$ that decreases to zero as $N \to \infty$.

$$\lim_{N \to \infty} E_N = \frac{1}{2L} \int_{-L}^{L} [f(x)]^2 \, dx - \frac{1}{4} a_0^2 - \frac{1}{2} \sum_{n=1}^{N} (a_n^2 + b_n^2) = 0.$$

Y. K. Goh
Fourier Series and Sturm-Liouville Eigenvalue Problems
Bessel’s Inequality and Parseval’s Identity

Since $E_N > 0$, from the definition of $E_N$, we get

\[
\frac{1}{4}a_0^2 + \frac{1}{2} \sum_{n=1}^{N} (a_n^2 + b_n^2) \leq \frac{1}{2L} \int_{-L}^{L} [f(x)]^2 \, dx.
\]

A stronger result is when taking the limit $N \rightarrow \infty$

\[
\frac{1}{4}a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{2L} \int_{-L}^{L} [f(x)]^2 \, dx.
\]
Multiplication Theorem

A generalisation of the Parseval’s identity is the multiplication theorem.

**Theorem (Multiplication (Inner Product) Theorem)**

If $f$ and $g$ are two $2L$-periodic piecewise smooth functions

$$
\frac{1}{2L} \int_{-L}^{L} f(x) g(x) \, dx = \sum_{n=-\infty}^{\infty} c_n d_n^*
$$

where $c_n$ and $d_n$ are the Fourier coefficients for the complex Fourier series of $f$ and $g$ respectively.
Complex Form of Fourier Series

Theorem (Complex Form of Fourier Series)

Let $f$ be a $2L$-periodic piecewise smooth function. The complex form of the Fourier series of $f$ is

$$
\sum_{n=-\infty}^{\infty} c_n e^{in\omega x},
$$

where the frequency $\omega = 2\pi/2L$ and the Fourier coefficients $c_n = \frac{1}{2L} \int_{-L}^{L} f(x)e^{-in\omega x} \, dx$, $n = 0, \pm 1, \pm 2, \ldots$.

For all $x$, the complex Fourier series converges to $f(x)$ if $f$ is continuous at $x$, and to $\frac{1}{2}[f(x+) + f(x-)]$ otherwise.
Relations of Complex and Real Fourier Coefficients

\[ c_{-n} = c_n^*; \]
\[ c_0 = \frac{1}{2} a_0 \]
\[ c_n = \frac{1}{2} (a_n - i b_n); \]
\[ c_{-n} = \frac{1}{2} (a_n + i b_n). \]

\[ a_0 = 2 c_0; \]
\[ a_n = c_n + c_{-n}; \]
\[ b_n = i (c_n - c_{-n}). \]
Theorem (Complex Form of Parseval’s Identity)

Suppose \( f \) is a square integrable \( 2L \)-periodic piecewise smooth function on \([-L, L]\). Then

\[
\frac{1}{2L} \int_{-L}^{L} [f(x)]^2 \, dx = \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2
\]

where \( c_n \) is the complex Fourier coefficients of \( f \).
Example

Find the complex Fourier series for the $2\pi$-periodic function $f(x) = e^{ix}$ defined in $(-\pi, \pi)$. 
The distribution of the magnitude of complex Fourier coefficients $|c_n|$ in frequency domain is called the amplitude spectrum of $f$.

The distribution of $p_0 = |c_0|^2$ and $p_n = |c_n|^2$ in frequency domain is called the power spectrum of $f$. 
Definition (Inner Products)

Let \( \psi \) and \( \phi \) be (possibly complex) functions of \( x \) on the interval \((a, b)\). Then the inner product of \( \psi \) and \( \phi \) is

\[
\langle \psi | \phi \rangle = \int_{a}^{b} \psi^*(x) \phi(x) \, dx.
\]

Note that the notation of inner product in some books is

\[
(\phi, \psi) = \int_{a}^{b} \psi^*(x) \phi(x) \, dx.
\]

Please take note on the order of \( \phi \) and \( \psi \) in the brackets.
**Definition (Norm)**

Let $f$ be (possibly complex) function of $x$ on the interval $(a, b)$. Then the norm of $f$ is

$$
||\psi|| = \sqrt{\langle f | f \rangle} = \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2}.
$$
Orthogonal Functions

Definition (Orthogonal Functions)

The functions $f$ and $g$ are called orthogonal on the interval $(a, b)$ if their inner product is zero,

$$\langle f | g \rangle = \int_{a}^{b} f^*(x)g(x) \, dx = 0.$$ 

Definition (Orthogonal Set of Functions)

A set of functions $\{F_1, F_2, F_3, \ldots \}$ defined on the interval $(a, b)$ is called an orthogonal set if

- $\|F_n\| \neq 0$ for all $n$; and
- $\langle F_m | F_n \rangle = 0$, for $m \neq n$. 

Y. K. Goh

*Fourier Series and Sturm-Liouville Eigenvalue Problems*
Normalisation and Orthonormal Set

Definition (Normalisation)
A normalised function $f_n$ for a function $F_n$, with $\|F_n\| \neq 0$, is defined as

$$f_n(x) = \frac{F_n(x)}{\|F_n\|}.$$

Definition (Orthonormal Set of Functions)
A set of functions $\{f_1, f_2, f_3, \ldots\}$ defined on the interval $(a, b)$ is called an orthonormal set if

- $\|f_n\| = 1$ for all $n$; and
- $\langle f_m | f_n \rangle = 0$, for $m \neq n$;

or simply, $\langle f_m | f_n \rangle = \delta_{mn}$, where $\delta_{mn}$ is the Kronecker’s delta.
Generalized Fourier Series

Theorem (Generalized Fourier Series)

If \( \{ f_1, f_2, f_3, \ldots \} \) is a complete set of orthogonal functions on \((a, b)\) and if \( f \) can be represented as a linear combination of \( f_n \), then the generalised Fourier series of \( f \) is given by

\[
f(x) = \sum_{n=1}^{\infty} a_n f_n(x) = \sum_{n=1}^{\infty} \frac{\langle f_n | f \rangle}{\| f_n \|^2} f_n(x),
\]

where \( a_n = \frac{\langle f_n | f \rangle}{\| f_n \|^2} \) is the generalised Fourier coefficient.
Theorem (Generalized Parseval’s Identity)

If \( \{f_1, f_2, f_3, \ldots \} \) is a complete set of orthogonal functions on \((a, b)\) and let \( f \) be such that \( \|f\| \) is finite. Then

\[
\int_{a}^{b} |f(x)|^2 \, dx = \sum_{n=1}^{\infty} \frac{|\langle f_n | f \rangle|^2}{\|f_n\|^2}.
\]

Y. K. Goh

Fourier Series and Sturm-Liouville Eigenvalue Problems
Orthogonality with respect to a Weight, $w(x)$

- (Inner product)
  $$\langle f \mid g \rangle = \int_a^b f^*(x)g(x)w(x) \, dx$$

- (Orthogonality)
  $$\langle f_m \mid f_n \rangle = \int_a^b f_m^*(x)f_n(x)w(x) \, dx = \|f_m\|^2 \delta_{mn}$$

- (Generalised Fourier Series)
  $$f(x) = \sum_{n=1}^{\infty} \frac{\langle f_n \mid f \rangle}{\|f_n\|^2} f_n(x)$$

- (Generalised Parseval’s Identity)
  $$\int_a^b |f(x)|^2w(x) \, dx = \sum_{n=1}^{\infty} \frac{|\langle f_n \mid f \rangle|^2}{\|f_n\|^2}$$
Adjoint and Self-adjoint Operators

Definition (Adjoints of Differential Operators)
Suppose $u$ and $v$ are (possibly complex) functions of $x$ in $(a, b)$ and let $L$ be a linear differential operator. Then, the formal adjoint $M$ of $L$ is another operator such that for all $u$ and $v$

$$\langle f | L[g] \rangle = \langle M[f] | g \rangle.$$ 

Definition (Self-adjoint Operators)
Suppose $M$ is the formal adjoint operator for a linear operator $L$ in space $S$. If $M = L$, then the operator $L$ is said to be formally self-adjoint or formally Hermitian.
Example: Adjoint for $L[\phi] \equiv p(x)\frac{d\phi}{dx}$

Suppose $L[\phi] \equiv p(x)d\phi/dx$, then

$$\langle u|L[v]\rangle = \int_a^b u^* \left[p \frac{d}{dx}\right] v \, dx = [u^*pv]_a^b - \int_a^b v \left[\frac{d}{dx}(p^*u)\right]^* \, dx$$

$$= \langle M[u]|v\rangle$$

In the last step, we set the boundary term to zero. The adjoint for the operator $L$ consists of

- Formal adjoint $M[\phi] \equiv -\frac{d}{dx}(p^*\phi)$; and
- Boundary conditions $[u^*pv]_a^b = 0$.

Furthermore, if $p(x)$ is a pure imaginary constant, then $M = L$. i.e. $L$ is self-adjoint.
Suppose $L[\phi] \equiv a_0(x)\phi'' + a_1(x)\phi' + a_2(x)\phi$. Then,

$$
\langle u | L[v] \rangle = [u^* a_0 v' + u^* a_1 v - v(a_0 u^*)']_a^b + \\
\int_a^b v [(a_0 u^*)'' - (a_1 u^*)' + a_2 u^*] \, dx = \langle M[u] | v \rangle \text{ with appropriate choice of boundary conditions. Note that}
$$

- $M[\phi] \equiv a_0 \phi'' + (2a'_0 - a_1)\phi' + (a_2 - a'_1 + a''_0)\phi$.
- $M$ can be made self-adjoint if $a_1 = a'_0$.
- The self-adjoint operator $S$ is
  $$
  S[\phi] = \frac{d}{dx} \left( a_0(x) \frac{d\phi}{dx} \right) + a_2(x)\phi.
  $$
- The neccessary boundary condition is
  $$
  [a_0 u^* v' - a_0 v (u^*)']_a^b = 0.
  $$
Definition (Eigenvalue Problem)

The eigenvalue problem associated to a differential operator $L$ is the equation $Ly + \lambda y = 0$, where $\lambda$ is called the eigenvalue, and $y$ is called the eigenfunction.

It is possible to find a weight factor $w(x) > 0$ for $L$ such that $S[y] \equiv w(x) L[y]$ is self-adjoint. The resulting eigenvalue equation is called the Sturm-Liouville Equation

Definition (Sturm-Liouville Equation)

$$[S + \lambda w(x)] y = \frac{d}{dx} \left( a_0(x) \frac{dy}{dx} \right) + a_2(x)y + \lambda w(x)y = 0 \text{ for } a < x < b.$$
Definition (Regular Sturm-Liouville Problem)

A regular SL problem is a boundary value problem on a closed finite interval \([a, b]\) of the form

\[
\frac{d}{dx} \left( a_0(x) \frac{dy}{dx} \right) + a_2(x)y + \lambda w(x)y = 0, \quad a < x < b,
\]

satisfying regularity conditions and boundary conditions

\[
c_1 y(a) + c_2 y'(a) = 0, \quad d_1 y(b) + d_2 y'(b) = 0,
\]

where at least one of \(c_1\) and \(c_2\) and at least one of \(d_1\) and \(d_2\) are non-zero.
Singular Sturm-Liouville Problems

Definition (Regularity Conditions)

The regularity conditions of a regular SL problem are

- \( a_0(x), a'_0(x), a_2(x) \) and \( w(x) \) are continuous in \([a, b]\);
- \( a_0(x) > 0 \) and \( w(x) > 0 \).

Definition (Singular Sturm-Liouville Problem)

A singular SL problem is a boundary value problem consists of Sturm-Liouville equation, but either

- fails the regularity conditions; or
- infinite boundary conditions; or
- one or more of the coefficients become singular.
A trivial (not useful) solution to the SL problem is $y = 0$.

Other non-trivial solutions would be the eigenfunctions $y_m$, and for each of these eigenfunctions there is a corresponding eigenvalue $\lambda_m$.

There are infinite many of these eigenfunctions, and the set of eigenfunctions $\{y_1, y_2, \ldots, y_m, \ldots\}$ forms a complete orthogonal set of functions that span the infinite dimensional Hilbert space.

Any function $f$ in the Hilbert space can be expressed as a linear combination of the eigenfunctions,

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x).$$
Theorem (Sturm-Liouville Problem)

The eigenvalues and eigenfunctions of a SL problem has the properties of

- All eigenvalues are real and compose a countably infinite collections satisfying $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ where $\lambda_j \to \infty$ as $j \to \infty$.

- To each eigenvalue $\lambda_j$ there corresponds only to one independent eigenfunction $y_j(x)$.

- The eigenfunctions $y_j(x)$, $j = 1, 2, \ldots$, compose a complete orthogonal set with appropriate to the weight functions $w(x)$ in doubly-integrable functions space $L^2(a, b)$. 

Y. K. Goh

Fourier Series and Sturm-Liouville Eigenvalue Problems
Eigenfunction Expansions

Theorem (Eigenfunction Expansions)

If \( f \in \mathcal{L}^2(a, b) \) then eigenfunction expansion of \( f \) on \( \{y_1, y_2, \ldots \} \) is

\[
f(x) = \sum_{n=1}^{\infty} A_n y_n, \quad a < x < b,
\]

where

\[
A_n = \frac{\langle y_n | f \rangle}{\|y_n\|^2} = \frac{\int_a^b y_n^*(x) f(x) w(x) \, dx}{\int_a^b |y_n(x)|^2 w(x) \, dx}.
\]
An example of SL equation is the Harmonic Equation with boundary condition.

- **ODE**: \( y'' + \lambda y = 0, \ 0 < x < L \).
- **Dirichlet Boundary condition**: \( y(0) = y(L) = 0 \).
- **Eigenvalues**: \( \lambda_n = k_n^2 = \frac{n^2\pi^2}{L^2}, \ n = 1, 2, \ldots \);
- **Eigenfunctions**: \( y_n(x) = \sin \left( \frac{n\pi}{L} x \right) \) for \( n = 1, 2, \ldots \);
- **Eigenfunction expansion**: \( f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \).
Example SL Problem: Harmonic Equation

Again, the Harmonic Equation with another boundary condition.

- **ODE**: $y'' + \lambda y = 0$, $0 < x < L$.
- **Neumann Boundary condition**: $y'(0) = y'(L) = 0$.
- **Eigenvalues**: $\lambda_n = k_n^2 = \frac{n^2 \pi^2}{L^2}$;
- **Eigenfunctions**: $y_n(x) = \cos\left(\frac{n\pi}{L} x\right)$ for $n = 0, 1, 2, \ldots$;
- **Eigenfunction expansion**: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right)$. 

Y. K. Goh

Fourier Series and Sturm-Liouville Eigenvalue Problems
Example SL Problem: Harmonic Equation

The Harmonic Equation with periodic boundary condition.

- **ODE:** \( y'' + \lambda y = 0, \ 0 < x < 2\pi. \)
- **Periodic boundary condition:** \( y(0) = y(2\pi). \)
- **Eigenvalues:** \( \lambda_n = k_n^2 = n^2; \)
- **Eigenfunctions:** \( y_n(x) = e^{inx} \) for \( n = 0, \pm 1, \pm 2, \ldots; \)
- **Eigenfunction expansion:** \( f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \)
Example SL Problem: (Parametric) Bessel Equation

Bessel Equation. \( x^2 y'' + xy' + (\lambda^2 x^2 - \mu^2) y = 0 \) or

\([x^2 y']' + (\lambda^2 x - \frac{\mu^2}{x})y = 0 \) in the interval \( 0 < x < L \).

- Since \( a_0(x) = x^2 \) is zero at \( x = 0 \), \( \implies \) singular SL problem.
- ODE: \( x^2 y'' + xy' + (\lambda^2 x^2 - \mu^2) y = 0 \);
- Boundary conditions: \( y(x) \) is bounded, and \( y(L) = 0 \);
- Eigenvalues: \( \lambda^2 = \lambda_n^2 = \frac{\alpha_n}{L} \), \( n = 1, 2, \ldots \), where \( \alpha_n \) is the \( n \)th-root of \( J_\mu \), i.e. \( J_\mu(\alpha_n) = 0 \);
- Eigenfunctions: \( y_n(x) = J_\mu(\lambda_n x) \), \( n = 1, 2, \ldots \);
- Weight: \( w(x) = x \);
- Eigenfunction expansion: \( f(x) = \sum_{n=1}^{\infty} a_n J_\mu(\lambda_n x) \).
Example SL Problem: Spherical Bessel Equation

Bessel Equation. \( x^2 y'' + xy' + (\lambda^2 x^2 - n(n + 1))y = 0 \) in the interval \( 0 < x < \infty \).

- Since \( a_0(x) = x^2 \) is zero at \( x = 0 \), \( \Rightarrow \) singular SL problem.
- ODE: \( x^2 y'' + xy' + (\lambda^2 x^2 - \mu(\mu + 1))y = 0 \);
- Boundary conditions: \( y(x) \) is bounded, and \( y(L) = 0 \);
- Eigenvalues: \( \lambda^2 = \lambda_n^2 = \frac{\alpha_n}{L} \), \( n = 1, 2, \ldots \), where \( \alpha_n \) is the \( n \)th-root of \( j_\mu \), i.e. \( j_\mu(\alpha_n) = 0 \);
- Eigenfunctions: \( y_n(x) = j_\mu(\lambda_n x) \), \( n = 1, 2, \ldots \);
- Weight: \( w(x) = x \);
- Eigenfunction expansion: \( f(x) = \sum_{n=1}^{\infty} a_n j_\mu(\lambda_n x) \).
Example SL Problem: Legendre Equation

Legendre Equation. \((1 - x^2)y'' - 2xy' + n(n + 1)y = 0\) or \([(1 - x^2)y']' + n(n + 1)y = 0\), in the interval \(-1 < x < 1\).

- \(a_0(x) = (1 - x^2)\) and \(a_0(\pm 1) = 0 \implies\) singular SL problem.
- ODE: \((1 - x^2)y'' - 2xy' + \lambda y = 0\);
- Boundary conditions: \(y(x)\) is bounded at \(x = \pm 1\);
- Eigenvalues: \(\lambda = \lambda_n = n(n + 1), \ n = 0, 1, 2, \ldots\);
- Eigenfunctions: \(y_n(x) = P_n(x), \ n = 0, 1, 2, \ldots\);
- Eigenfunction expansion: \(f(x) = \sum_{n=0}^{\infty} a_n P_n(x)\).