# Boundary Value Problems in Cylindrical Coordinates

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# Outline

- Differential Operators in Various Coordinate Systems
- Laplace Equation in Cylindrical Coordinates Systems
- Bessel Functions
- Wave Equation the Vibrating Drumhead
- Heat Flow in the Infinite Cylinder
- Heat Flow in the Finite Cylinder

# Differential Operators in Various Coordinates

# Differential Operators in Polar Coordinates

- Polar Coordinates
  - $x = r\cos\theta, \quad y = r\sin\theta$
  - $r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$
- Differential relations
  - $\frac{\partial r}{\partial x} = \frac{x}{r}, \ \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}$, and $\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}, \ \frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{r^4}$
  - $\frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$, and $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}, \ \frac{\partial^2 \theta}{\partial y^2} = -\frac{2xy}{r^4}$
  - $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = 0$, and $\dfrac{\partial \theta}{\partial x}\dfrac{\partial r}{\partial x} + \dfrac{\partial \theta}{\partial y}\dfrac{\partial r}{\partial y} = 0$
- Differential operators
  - $\frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$, and $\frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$
  - $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$

# Laplacian in Various Coordinates

- Polar Coordinates
  - $(r, \theta) : x = r\cos\theta,\ y = r\sin\theta$
  - $\nabla^2 u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$
- Cylindrical Coordinates
  - $(\rho, \phi, z) : x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$
  - $\nabla^2 u = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho\dfrac{\partial u}{\partial \rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\partial^2 u}{\partial z^2}$
- Spherical Coordinates
  - $(r, \theta, \phi) : x = r\cos\phi\sin\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\theta$
  - $\nabla^2 u =$
    $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial u}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 u}{\partial \phi^2}$

# Laplace Equation on Cylindrical Coordinates

# Dirichlet Problems on a Disk or inside Inf Cylinder

A Dirichlet's problem inside a Disk or Infinite Cylinder

- Variables $u(\rho, \phi, z) = u(r, \phi)$.
- PDE

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 0,\ 0 < r < L,\ 0 \leq \phi < 2\pi.$$

- Periodic Boundary Conditions
  - $u(r, \phi) = u(r, \phi + 2\pi), \quad 0 < r < L,\ 0 \leq \phi < 2\pi;$
  - $u(L, \phi) = f(\phi), \quad 0 \leq \phi < 2\pi.$
- Regularity Conditions: $|u|$ is finite in $0 < r < L$.

# Dirichlet Problems on a Disk or inside Inf Cylinder

- Separation of variables
$$u(r,\phi) = R(r)\Phi(\phi) \implies \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{\Phi''}{\Phi} = \lambda.$$
- Sturm-Liouville Eq. associated to $\Phi$ is Harmonic Eq.
$$\begin{cases} \Phi'' + \lambda\Phi = 0, & 0 \le \phi < 2\pi, \\ \Phi(0) = \Phi(2\pi), \\ \Phi'(0) = \Phi'(2\pi). \end{cases}$$
- Eigenvalue and eigenfunction for the Sturm-Liouville Eq.
$$\lambda_n = n^2, n = 0, 1, 2, \ldots$$
$$\Phi_n(\phi) = \begin{cases} A_0, & n = 0; \\ A_n \cos n\phi + B_n \sin n\phi, & n = 1, 2, \ldots. \end{cases}$$

# Dirichlet Problems on a Disk or inside Inf Cylinder

- ODE associated to $R$ is a Euler Equation

$$r^2 R'' + rR' - \lambda R = 0, \quad 0 < r < L.$$

$$R(r) = \begin{cases} C_0 + D_0 \ln r & \text{for } \lambda = 0; \\ C_n r^n + D_n r^{-n}, & \text{for } \lambda_n \neq 0, n = 1, 2, \ldots. \end{cases}$$

- Regularity condition: $|u|$ is finite in $0 < r < L$ requires that $D_0 = D_n = 0$, because $\ln r \to -\infty$ and $r^{-n} \to \infty$ as $r \to 0$.
- A general solution $0 < r < L$ and $0 \leq \phi < 2\pi$ is

$$u(r, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n r^n \cos n\phi + b_n r^n \sin n\phi), \; n = 1, 2, \ldots$$

# Dirichlet Problems on a Disk or inside Inf Cylinder

- Coefficients
  - $a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi$
  - $a_n L^n = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \cos n\phi\, d\phi$
  - $b_n L^n = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \sin n\phi\, d\phi$
- A complex form of the general solution is
  - $u(r, \phi) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\phi}$, $0 < r < L$, $0 \leq \phi < 2\pi$.
  - $c_0 L^n = \frac{1}{2\pi} \int_0^{\infty} f(\phi) e^{-in\phi}\, d\phi, \quad n = 0, \pm 1, \pm 2, \ldots$

# Neumann Problems on a Disk or inside Inf Cylinder

A Neumann's problem on a Disk or inside Infinite Cylinder

- Variables $u(\rho, \phi, z) = u(r, \phi)$.
- PDE

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 0,\ 0 < r < L,\ 0 \le \phi < 2\pi.$$

- Periodic Boundary Conditions
  - $u(r, \phi) = u(r, \phi + 2\pi), \quad 0 < r > L,\ 0 \le \phi < 2\pi;$
  - $\frac{\partial u}{\partial r}(L, \phi) = f(\phi), \quad 0 \le \phi < 2\pi.$
- Regularity Conditions: $|u|$ is finite in $0 < r < L$.

# Neumann Problems on a Disk or inside Inf Cylinder

- General solution
$$u(r,\phi) = \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n r^n \cos n\phi + b_n r^n \sin n\phi)$$
- Boundary condition
$$\left.\frac{\partial u}{\partial r}\right|_{r=L} = \sum_{n=1}^{\infty}\left(a_n n L^{n-1}\cos n\phi + b_n n L^{n-1}\sin n\phi\right) = f(\theta)$$
- Coefficients
  - $a_n n L^{n-1} = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\cos n\phi\, d\phi$
  - $b_n n L^{n-1} = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\sin n\phi\, d\phi$
- Note that the B.C. **must** satisfies $\int_0^{2\pi} f(\phi)\, d\phi = 0$

# Dirichlet Problems outside a Disk or Inf. Cylinder

A Dirichlet's problem outside a Disk or Infinite Cylinder

- Variables $u(\rho, \phi, z) = u(r, \phi)$.
- PDE

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 0,\ r > L,\ 0 \le \phi < 2\pi.$$

- Periodic Boundary Conditions
  - $u(r, \phi) = u(r, \phi + 2\pi), \quad r > L,\ 0 \le \phi < 2\pi;$
  - $u(L, \phi) = f(\phi), \quad 0 \le \phi < 2\pi.$
- Regularity Conditions $u$ and $\int_L^\infty |u|\, dr$ is finite for $r > L$ and $u \to 0$ as $r \to \infty$.

# Dirichlet Problems outside a Disk or Inf. Cylinder

- Similar to the Laplace equation inside a disk:
  - Eigenvalue: $\lambda = n^2$, $n = 0, 1, 2, \ldots$
  - Eigenfunction:
    $$\Phi_n(\phi) = \begin{cases} A_0, & n = 0; \\ A_n \cos n\phi + B_n \sin n\phi, & n = 1, 2, \ldots. \end{cases}$$
  - Solution for Euler Eq:
    $$R(r) = \begin{cases} C_0 + D_0 \ln r & \text{for } \lambda = 0; \\ C_n r^n + D_n r^{-n}, & \text{for } \lambda_n \neq 0, n = 1, 2, \ldots, \end{cases}$$
    and $D_0 = C_n = 0$ as $R(r)$ must be finite when $r \to \infty$.
- General solution
  $$u(r, \phi) = \frac{a_0}{2} + \sum_{n=0}^{\infty} (a_n r^{-n} \cos n\phi + b_n r^{-n} \sin n\phi)$$

# Neumann Problems outside a Disk or Inf Cylinder

A Neumann's problem outside a Disk or Infinite Cylinder

- Variables $u(\rho, \phi, z) = u(r, \phi)$.
- PDE

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 0,\ r > L,\ 0 \le \phi < 2\pi.$$

- Periodic Boundary Conditions
  - $u(r, \phi) = u(r, \phi + 2\pi), \quad r > L,\ 0 \le \phi < 2\pi;$
  - $\frac{\partial u}{\partial r}(L, \phi) = f(\phi), \quad 0 \le \phi < 2\pi.$
- Regularity Conditions $u$ and $\int_L^\infty |u|\, dr$ is finite for $r > L$ and $u \to 0$ as $r \to \infty$.

# Neumann Problem outside a Disk or Inf Cylinder

- General solution
$$u(r,\phi)=\frac{a_0}{2}+\sum_{n=0}^{\infty}(a_n r^{-n}\cos n\phi+b_n r^{-n}\sin n\phi)$$
- Boundary condition
$$\left.\frac{\partial u}{\partial r}\right|_{r=L}=\sum_{n=1}^{\infty}\left(-a_n nL^{-n-1}\cos n\phi-b_n nL^{-n-1}\sin n\phi\right)=f(\phi)$$
- Coefficients
  - $-a_n nL^{-n-1}=\frac{1}{2\pi}\int_0^{2\pi}f(\phi)\cos n\phi\,d\phi$
  - $-b_n nL^{-n-1}=\frac{1}{2\pi}\int_0^{2\pi}f(\phi)\sin n\phi\,d\phi$
- Note that the B.C. **must** satisfies $\int_0^{2\pi}f(\phi)\,d\phi=0$

# Dirichlet Problem on a Ring

A Dirichlet's problem on a ring

- Variables $u(\rho, \phi, z) = u(r, \phi)$.
- PDE + B.C.
  - $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \phi^2} = 0,\ A < r < B,\ 0 \le \phi < 2\pi.$
  - $u(r, \phi) = u(r, \phi + 2\pi), \quad A < r < B,\ 0 \le \phi < 2\pi;$
  - $u(A, \phi) = f_A(\phi),\ u(B, \phi) = f_B(\phi), \quad 0 \le \phi < 2\pi.$
- General solution

$$u(r, \phi) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left\{ (a_n r^n + b_n r^{-n}) \cos n\phi + (a_n r^n + b_n r^{-n}) \sin n\phi \right\}$$

# Robin's Problem on a Wedge

A Robin's problem on a wedge

- Variables $u(\rho, \phi, z) = u(r, \phi)$.
- PDE + B.C..
  - $\nabla^2 u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \phi^2} = 0,\ 0 < r < L,\ 0 \leq \phi < \alpha.$
  - $u(r, 0) = u(r, \alpha) = 0, \quad 0 < r < L;$
  - $\frac{\partial}{\partial r} u(L, \phi) = -u(L, \phi) - \phi, \quad 0 \leq \phi < \alpha.$
- Solution $u(r, \phi) = \displaystyle\sum_{n=1}^{\infty} b_n r^{\frac{n\pi}{\alpha}} \sin \frac{n\pi}{\alpha}\phi$ where

$$b_n = \frac{2\alpha^2(-1)^n}{n\pi L(\alpha + n\pi)}$$

# Bessel Functions

Bessel's functions have wide application in mathematics, especially when dealing with cylindrical symmetry or cylindrical coordinate systems.

## Definition (Bessel's functions & Bessel's equation)

Bessel's functions $J_\nu$ or $Y_\nu$ are solutions to the Bessel's equation of order $\nu$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0. \tag{1}$$

# Bessel Functions

- If we seek a series solution about $x = 0$ for the Bessel's equation, then

$$y(x) = \sum_{n=-\infty}^{\infty} a_n x^{n+r}, \quad a_n = 0 \text{ for } n < 0,$$

as $x = 0$ is a regular singular point.

- Substitute into the Bessel's equation, we obtain
  - Indicial equation $r^2 - \nu^2 = 0, \implies r = \pm\nu$
  - $r_1 - r_2 = 2\nu$, and if $2\nu$ is integer, we need to find another linearly independent series solution
  - Recurrence relation: $a_1 = 0$, and $n(n \pm 2\nu)a_n = -a_{n-2}$, $n \geq 2$.

# Bessel Functions

If $2\nu$ is **not** an integer,

- two linearly independent solutions for Eq.(1)are
  - $y_1(x) = x^\nu \left[1 - \frac{x^2}{2(2+2\nu)} + \frac{x^4}{2(4)(2+2\nu)(4+2\nu)} + \dots\right]$
  - $y_2(x) = x^\nu \left[1 - \frac{x^2}{2(2-2\nu)} + \frac{x^4}{2(4)(2-2\nu)(4-2\nu)} + \dots\right]$
- After normalizing the solutions, we obtain
  - the **general solution** to the Bessel's equation $y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$, where
  - $J_\nu(x) \propto y_1(x)$; and
  - $J_{-\nu}(x) \propto y_2(x)$; and
- $J_\nu(x)$ is called the **Bessel's functions of the first kind**.

# Bessel Functions

## Definition (Bessel's functions of the first kind)

The Bessel's function of the first kind of order $\nu$ is defined as

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{\nu+2n}}{n!(n+\nu)!}$$

$$= \frac{1}{\nu!}(\frac{1}{2}x)^\nu \left[1 - \frac{\left(\frac{x}{2}\right)^2}{\nu+1} + \frac{\left(\frac{x}{2}\right)^4}{2(\nu+1)(\nu+2)} + \dots\right].$$

## Definition (Gamma/factorial function)

For $\nu$ is not an integer, $\nu!$ is defined as

$$\nu! \equiv \Gamma(\nu+1) \equiv \int_0^\infty t^\nu e^{-t}\, dt.$$

# Bessel Functions

If $2\nu$ **is** an integer, and

- $\nu = N + \frac{1}{2}$, for some integer $N \geq 0$,
  - the resulting functions are called spherical Bessel's functions
  - $j_N(x) = (\pi/2x)^{1/2} J_{N+1/2}(x)$
  - (We will come back to speherical Bessel's function later)
- $\nu = N$, for some integer $N \geq 0$,
  - $J_N(x) = (-1)^N J_N(x)$ is linearly dependent to $J_N(x)$;
  - a linearly independent second solution is $Y_N(x)$ and it is called the **Bessel's function of the second kind**; now
  - the **general solution** of Eq.(1) is $y(x) = c_1 J_N(x) + c_2 Y_N(x)$.

# Bessel Functions

### Definition (Bessel's functions of the second kind)

The Bessel's function of the second kind of order $\nu$ is defined as

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}.$$

If $\nu = N$, for some integer $N \geq 0$, then

$$Y_N(x) = \lim_{\nu \to N} Y_\nu(x).$$

# Bessel Functions

- A variation of Bessel's equation of order $\nu$ is of the form,

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \tag{2}$$

- Eq.(2) can be converted to the standard Bessel's Equation by defining new variable $t = \lambda x$.
- The general solution is

$$y(x) = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x).$$

# Bessel Functions

A **modified Bessel's Equation** is of the form

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0. \qquad (3)$$

The general solution to modified Bessel's equation is

$$\begin{aligned} y(x) &= c_1 J_\nu(ix) + c_2 Y_\nu(ix); \quad \text{or} \\ y(x) &= c_1 I_\nu(x) + c_2 K_\nu(x). \end{aligned}$$

## Definition (Modified Bessel's functions)

- First kind $I_\nu(x) = i^{-\nu} J_\nu(ix) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{\nu+2n}}{n!(n+\nu)!}$;
- Second kind $K_\nu(x) = \lim_{r\to\nu} \frac{I_r(x)\cos r\pi - I_{-r}(x)}{\sin r\pi}$.

# Bessel Functions

A variation of modified Bessel's Equation is of the form

$$x^2y'' + xy' - (\lambda^2x^2 + \nu^2)y = 0, \tag{4}$$

with the general solution

$$y(x) = c_1 I_\nu(\lambda x) + c_2 K_\nu(\lambda x).$$

# Bessel Functions

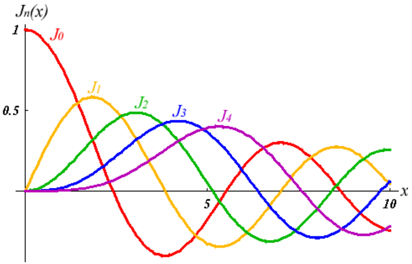


Figure: Bessel's functions of the first kind, $J_n(x)$.

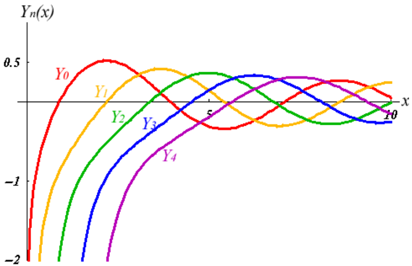


Figure: Bessel's functions of the second kind, $Y_n(x)$.

# Bessel Functions

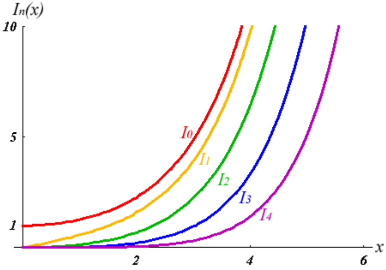


Figure: Modified Bessel's functions of the first kind, $I_n(x)$.

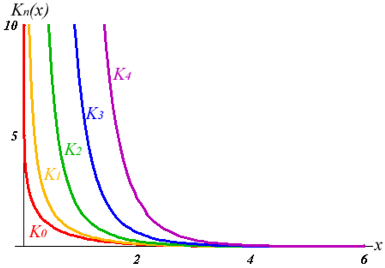


Figure: Modified Bessel's functions of the second kind, $K_n(x)$.

# Bessel Functions

Another representation of the Bessel's functions for integer values of $\nu = N$ is in the integral form

- $J_N(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(N\tau - x\sin\tau)\,d\tau$
- $Y_N(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \sin(x\sin\tau - N\tau)\,d\tau - \dfrac{1}{\pi}\displaystyle\int_0^\infty \left[e^{N\tau} + (-1)^N e^{-N\tau}\right] e^{-x\sinh\tau}\,d\tau$

# Bessel Functions

Also the Bessel's functions could be generated from the generating function

- $$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$
- $$e^{iz\cos\phi} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\phi}$$

# Bessel Functions

Recursion relations of Bessel's functions

- $\dfrac{d}{dx}[x^{\nu} J_{\nu}(x)] = x^{\nu} J_{\nu-1}(x)$
- $\dfrac{d}{dx}[x^{-\nu} J_{\nu}(x)] = -x^{-\nu} J_{\nu+1}(x)$
- $J_{\nu-1}(x) + J_{\nu+1}(x) = \dfrac{2\nu}{x} J_{\nu}(x)$
- $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_{\nu}(x)$
- $J'_{\nu}(x) = -\dfrac{\nu}{x} J_{\nu}(x) + J_{\nu-1}(x) = \dfrac{\nu}{x} J_{\nu}(x) - J_{\nu+1}(x)$

# Bessel Functions

Orthogonality properties of Bessel's functions:

- If $\alpha$ and $\beta$ are two roots of $J_\nu$, then

$$\begin{aligned}\int_0^1 x J_\nu(\alpha x) J_\nu(\beta x)\, dx &= \frac{\delta_{\alpha,\beta}}{2} J_{\nu+1}^2(\alpha) \\ &= \frac{\delta_{\alpha,\beta}}{2} J_\nu'^2(\alpha)\end{aligned}$$

- Thus, the generalized Fourier series in terms of $J_\nu$ is $f(x) = \sum_{n=1}^{\infty} c_n J_\nu(\lambda_n x)$, and where the coefficient is given by $c_n = \frac{\int_0^1 x f(x) J_\nu(\lambda_n x)\, dx}{\int_0^1 x J_\nu^2(\lambda_n x)\, dx} = \frac{2}{J_{\nu+1}^2(\lambda_n)} \int_0^1 x f(x) J_\nu(\lambda_n x)\, dx.$

# The Vibrating Drumhead (Radial Symmetric I.C.)

The vibrating of a thin circular membrane with uniform density with radial symmetric initial conditions is given by

- PDE: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \left( \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} \right), \quad 0 < r < \ell,\ t > 0.$
- B.C.: $u(\ell, t) = 0, \quad t \geq 0;$
- I.C.: $u(r, 0) = f(r), \quad \dfrac{\partial u}{\partial r}(r, 0) = g(r), \quad 0 < r < \ell.$
- The initial conditions are said to be **radial symmetric** as $f$ and $g$ are depend only on $r$ but not $\phi$.

# The Vibrating Drumhead (Radial Symmetric I.C.)

- Separation of variables:
$$u(r,t) = R(r)T(t) \implies \frac{T''}{c^2T} = \frac{1}{R}(R'' + \frac{1}{r}R') = -\lambda^2.$$
- The ODEs and corresponding B.C.

$$rR'' + R' + \lambda^2 rR = 0, \quad R(\ell) = 0 \tag{5}$$
$$T'' + c^2\lambda^2 T = 0 \tag{6}$$

- Solutions to ODEs $R(r) = c_1 J_0(\lambda r) + Y_0(\lambda r)$
  - Since $u$ is bounded near $r = 0$, this gives $c_2 = 0$;
  - B.C. $R(\ell) = 0 \implies J_0(\lambda\ell) = 0$;
  - Let $\alpha_n$, $n = 1, 2, \ldots$ be the roots of $J_0$, then $\lambda = \lambda_n = \frac{\alpha_n}{\ell}$, $n = 1, 2, \ldots$.

# The Vibrating Drumhead (Radial Symmetric I.C.)

- Non-trivial solutions to the ODEs

$$
\begin{aligned}
R_n(r) &= J_0\left(\lambda_n r\right) = J_0\left(\frac{\alpha_n}{\ell} r\right), n = 1, 2, \ldots, \\
T_n(t) &= A_n \cos c\lambda_n t + B_n \sin c\lambda_n t
\end{aligned}
$$

- General solution and coefficients

$$
\begin{aligned}
u(r,t) &= \sum_{n=1}^{\infty} (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) J_0\left(\lambda_n r\right), \\
A_n &= \frac{2}{\ell^2 J_1^2(\alpha_n)} \int_0^\ell f(r) J_0(\lambda_n r) r\, dr, \\
c\lambda_n B_n &= \frac{2}{\ell^2 J_1^2(\alpha_n)} \int_0^\ell g(r) J_0(\lambda_n r) r\, dr.
\end{aligned}
$$

# The Vibrating Drumhead (Asymmetric I.C.)

The vibrating of a thin circular membrane with uniform density with asymmetric initial conditions is given by

- PDE: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \left( \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \phi^2} \right),$
  $0 < r < \ell,\ 0 \leq \phi < 2\pi,\ t > 0.$
- B.C.: $u(\ell, \phi, t) = 0, \quad 0 \leq \phi < 2\pi,\ t \geq 0;$
- I.C.: $u(r, \phi, 0) = f(r, \phi), \quad \dfrac{\partial u}{\partial r}(r, \phi, 0) = g(r, \phi),$
  $0 < r < \ell,\ 0 \leq \phi < 2\pi.$
- The initial conditions are said to be **asymmetric** as $f$ and $g$ are depend on $r$ and $\phi$.

# The Vibrating Drumhead (Asymmetric I.C.)

- Separation of variables: $u(r,\phi,t) = R(r)\Phi(\phi)T(t) \implies \frac{T''}{c^2T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Phi''}{r^2\Phi} = -\lambda^2$; and $\frac{r^2R''+rR'+\lambda r^2 R}{R} = -\frac{\Phi''}{\Phi} = \mu^2$.
- The ODEs and corresponding B.C.

$$\Phi'' + \mu^2\Phi = 0, \quad \Phi(0) = \Phi(2\pi), \Phi'(0) = \Phi'(2\pi)$$
$$r^2R'' + rR' + (\lambda^2 r^2 - \mu^2)R = 0, \quad R(\ell) = 0$$
$$T'' + c^2\lambda^2 T = 0$$

- Solutions to ODEs $\lambda_{mn} = \frac{\alpha_{mn}}{\ell}$ and $\alpha_{mn}$ is $n^{\text{th}}$ root of $J_m$.

$$\Phi(\phi) = \Phi_m(\phi) = A_m \cos m\phi + B_m \sin m\phi,\ m = 0, 1, 2, \dots$$
$$R(r) = R_{mn}(r) = J_m(\lambda_{mn} r),\ m = 0, 1, 2, \dots, n = 1, 2, \dots$$
$$T(t) = T_{mn}(t) = A_{mn} \cos c\lambda_{mn} t + B_{mn} \sin c\lambda_{mn} t.$$

# The Vibrating Drumhead (Asymmetric I.C.)

General solution $u(r, \phi, t)$ and coefficients

$$
\begin{aligned}
u &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(a_{mn}\cos m\phi + b_{mn}\sin m\phi)\cos c\lambda_{mn} t \\
&+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(a^*_{mn}\cos m\phi + b^*_{mn}\sin m\phi)\sin c\lambda_{mn} t \\
a_{0n} &= \frac{1}{\pi \ell^2 J_1^2(\alpha_{0n})}\int_0^{\ell}\int_0^{2\pi} f(r,\phi) J_0(\lambda_{0n} r)\, r\, dr\, d\phi \\
a_{mn} &= \frac{2}{\pi \ell^2 J_{m+1}^2(\alpha_{mn})}\int_0^{\ell}\int_0^{2\pi} f(r,\phi)\cos m\phi J_m(\lambda_{mn} r)\, r\, dr\, d\phi \\
b_{mn} &= \frac{2}{\pi \ell^2 J_{m+1}^2(\alpha_{mn})}\int_0^{\ell}\int_0^{2\pi} f(r,\phi)\sin m\phi J_m(\lambda_{mn} r)\, r\, dr\, d\phi
\end{aligned}
$$

# The Vibrating Drumhead (Asymmetric I.C.)

$$
\begin{aligned}
a_{0n}^* &= \frac{1}{\pi \ell^2 J_1^2(\alpha_{0n})} \int_0^{\ell} \int_0^{2\pi} g(r,\phi) J_0(\lambda_{0n} r)\, r\, dr\, d\phi \\
a_{mn}^* &= \frac{2}{\pi \ell^2 J_{m+1}^2(\alpha_{mn})} \int_0^{\ell} \int_0^{2\pi} g(r,\phi) \cos m\phi J_m(\lambda_{mn} r)\, r\, dr\, d\phi \\
b_{mn}^* &= \frac{2}{\pi \ell^2 J_{m+1}^2(\alpha_{mn})} \int_0^{\ell} \int_0^{2\pi} g(r,\phi) \sin m\phi J_m(\lambda_{mn} r)\, r\, dr\, d\phi
\end{aligned}
$$

# Heat Flow in the Infinite Cylinder (Radial Sym.)

The radial flow of heat in a infinite cylinder or on a disk with lateral surface is kept at zero temperature is given by

- PDE: $\dfrac{\partial u}{\partial t} = c^2 \left( \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial u}{\partial r} \right), \, 0 < r < \ell, \, t > 0.$
- B.C.: $u(\ell, t) = 0, \quad t \geq 0;$
- I.C.: $u(r, 0) = f(r), \quad 0 < r < \ell.$
- The initial conditions are said to be **radial symmetric** as $f$ is depend only on $r$.

# Heat Flow in the Infinite Cylinder (Radial Sym.)

- Separation of variables:

$$u(r,t) = R(r)T(t) \implies \frac{T'}{c^2T} = \frac{1}{R}(R'' + \frac{1}{r}R') = -\lambda^2.$$

- The ODEs and corresponding B.C.

$$\begin{aligned} rR'' + R' + \lambda^2 rR &= 0, \quad R(\ell) = 0 \\ T' + c^2\lambda^2 T &= 0. \end{aligned}$$

- Solutions to ODEs

$$\begin{aligned} R_n(r) &= A_n J_0(\lambda_n r) \\ T_n(t) &= C_n e^{-c^2\lambda_n^2 t} \\ \lambda_n &= \alpha_n/\ell,\ n = 1, 2, \ldots, \quad \alpha_n \text{ are roots of } J_0. \end{aligned}$$

# Heat Flow in the Infinite Cylinder (Radial Sym.)

- General solution and coefficients

$$
\begin{aligned}
u(r,t) &= \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) e^{-c^2 \lambda_n^2 t}, \\
a_n &= \frac{2}{\ell^2 J_1^2(\alpha_n)} \int_0^{\ell} f(r) J_0(\lambda_n r) r \, dr.
\end{aligned}
$$

# Heat Flow in a Finite Cylinder

The heat flow in a finite cylinder of radius $\ell$ and height $2h$ with lateral surface and bases are kept at zero temperature is given by

- PDE: $\dfrac{\partial u}{\partial t} = c^2\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\partial^2 u}{\partial z^2}\right)$,
  $0 < r < \ell,\ 0 \leq \phi < 2\pi,\ -h < z < h,\ t > 0.$
- B.C.: $u(\ell, \phi, z, t) = u(r, \phi, -h, t) = u(r, \phi, h, t) = 0,$
- I.C.: $u(r, \phi, z, 0) = f(r, \phi, z).$

# Heat Flow in a Finite Cylinder

- Separation of variables: $u(r, \phi, z, t) = R(r)\Phi(\phi)Z(z)T(t)$
- The general solution

$$u(r, \phi, z, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left\{ e^{-c^2(\lambda_{mn}^2+\mu_k^2)t} J_m(\lambda_{mn} r) \sin \mu_k (z+h) \right. \left. (A_{mnk} \cos m\phi + B_{mnk} \sin m\phi) \right\}$$