UECM3033 Numerical Methods

Tutorial : Numerical Differentiation & Integration

(Jan 2013)

1. The integral $ERF(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$ is called the error function. It appears frequently in applied mathematics and statistics. Calculate $ERF(1)$ by using respectively:
   - the basic Trapezoid rule,
   - the composite Trapezoid rule with 16 subdivisions,
   - the basic Simpsons rule,
   - the composite Simpsons rule with 16 subdivisions.
   - Improve the accuracy of the Trapezoidal approximation by performing Romberg integration.
   - Compare you results with answer obtained from running MATLAB command `erf(1)`. What is the accuracy of your result?

2. Repeat the previous exercise with $\Gamma(\nu = 0.5) = \int_0^\infty t^\nu e^{-t}\, dt$. (Instead of letting $t \to \infty$ as upper limit, use $t = 10$ as upper limit.)

3. Evaluate the following integrals with composite Simpson's rule with 8 subdivisions and with the appropriate change of variable:
   - $\int_0^\infty e^{-x^2/2}\, dx$, using $x = -\ln t$
   - $\int_0^\infty x^{-1} \sin x\, dx$, using $x = t^{-1}$
   - $\int_0^\infty \sin x^2\, dx$, using $x = \tan t$

4. A Gaussian quadrature rule for the interval $[-1, 1]$ can be used on the interval $[a, b]$ by applying a suitable linear transformation. Approximate

$$\int_0^2 e^{-x^2}\, dx$$

   by applying suitable transformation for Gaussian quadrature with $n = 1$.

5. Construct a rule of the form

$$\int_{-1}^1 f(x)\, dx \approx a_1 f(-\tfrac{1}{2}) + a_2 f(0) + a_3 f(\tfrac{1}{2})$$

   that is exact for all polynomials of degree ≤ 2; i.e. determine the values of a_1, a_2 and a_3.

6. Derive from first principles, the three-point closed and equally spaced quadrature formula. That is, find the coefficients a_1, a_2 and a_3 such that

$$\int_a^b f(x)\, dx = h[a_1 f(a) + a_2 f(a+h) + a_3 f(b)]$$

for any polynomial $f(x)$ up to degree 2. Show that the formula also work for cubic polynomials as well.

7. Determine the error term for the formula

$$f'(x) \approx \frac{1}{4h}[f(x+3h) - f(x-h)].$$

8. Derive the approximation formula

$$f'(x) \approx \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)]$$

and show that its error term is of the form $\frac{1}{2}h^2 f'''(\xi)$.

9. Derive the formula for estimating $f'''(x)$,

$$f'''(x) \approx \frac{1}{2h^3}[f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)].$$

10. Estimate numerical $f'(\sqrt{2})$ and corresponding absolute error for $f(x) = \tan^{-1}(x)$ using
    a) Forward difference formula with $h = 0.01$.
    b) Backward difference formula with $h = 0.01$.
    c) Center difference formula with $h = 0.005$.

11. A certain calculation requires an approximation formula for $f'(x) + f''(x)$. Derive a approximation formula and determine its error term by using a forward difference scheme for $f'(x)$ and a center difference scheme for $f''(x)$. How do you improve the approximation of $f'(x) + f''(x)$?

12. Let $f(x) = \tan^{-1} x$. Use the center difference formula and Richardson's extrapolation method to find the approximations to $f'(1)$ accurate to the order of $O(h^2), O(h^4)$ and $O(h^6)$ with $h = 0.1$. Calculate the errors for each approximations. Write your answers in 10 significant digits.