UECM3033 Numerical Methods

Tutorial : Numerical ODE & Initial Value Problem

(Jan 2013)

1. Consider the initial value problem
$$y' = -(y+1)(y+3), \quad 0 \le t \le 1, \text{ with } y(0) = -2.$$
With a step size of $h = 0.2$, solve the IVP for $0 \le t \le 1$ by using the following algorithms:
   a) Euler's method
   b) Second order Taylor's scheme
   c) 4th order Taylor's scheme
   d) Runge-Kuttas method of order 2
   e) Runge-Kuttas method of order 4
   f) multi-step leadfrog algorithm

   Also solve the IVP analytically and for each of the method used, find the local errors on every mesh points.

2. Repeat Q1 with step size $h = 0.1$ the following IVP:
$$y' = y - 2x/y, \quad 0 \le t \le 0.5, \text{ with } y(0) = 1.$$

3. An engineer used Fourth-Order Runge-Kutta method to solve $y' = f(t, y)$ over the interval $[0, 15]$ with a step size of $h = 0.5$ obtaining $y(15) \approx 34.221$, and then repeated the computation with $h = 0.125$ obtaining $y(15) \approx 33.334$. Show how it is possible to combine these two estimates to get a better estimate for $y(15)$. What is the order of error in your estimate?

4. The irreversible chemical reaction in which two molecules of solid potassium dichromate ($K_2Cr_2O_7$), two molecules of water (H_2O), and three atoms of solid sulfur (S) combine to yield three molecules of the gas sulfur dioxide (SO_2), four molecules of solid potassium hydroxide (KOH), and two molecules of solid chromic oxide (Cr_2O_3) can be represented by the stoichiometric equation
$$2K_2Cr_2O_7 + 2H_2O + 3S \to 4KOH + 2Cr_2O_3 + 3SO_2.$$
If n_1 molecules of $K_2Cr_2O_7$, n_2 molecules of H_2O, and n_3 molecules of S are originally available, the following equation describes the amount $x(t)$ of KOH after time t:
$$\frac{dx}{dt} = k\left(n_1 - \frac{x}{2}\right)^2\left(n_2 - \frac{x}{2}\right)^2\left(n_3 - \frac{3x}{4}\right)^3$$
where $k = 6.22 \times 10^{-19}, n_1 = n_2 = 2 \times 10^3, n_3 = 3 \times 10^3$. Find the amount of potassium hydroxide formed after 0.2 s by using MatLab `ode45` equation solver. Plot the graph for $x(t)$ versus t.

5. Repeat Q1 with $h = 0.25$ for the following IVP in $[0, 1]$:

$$\begin{aligned} \frac{dy_1}{dt} &= 4y_2, \quad y_1(0) = 2 \\ \frac{dy_2}{dt} &= y_1, \quad y_2(0) = 0. \end{aligned}$$

6. Repeat Q1 with $h = 0.25$ for the following IVP in $[0, 1]$:

$$y'' - 3y' + 2y = x, \quad y(0) = 0, y'(0) = 1/2.$$

7. Consider the Runge-Kutta method of order 2,

$$\begin{aligned} k_1 &= hf(t_i, y_i) \\ k_2 &= hf(t_i + h/2, y_i + k_1/2) \\ y_{i+1} &= y_i + \frac{1}{4}k_1 + \frac{3}{4}k_2 \end{aligned}$$

   a) Find the absolute stability function for the initial value problem, $y' = -\lambda y, y(0) = y_0$.

   b) Find the maximum step size h allowed in order for the method to be stable, given $\lambda = 4$.