UECM3033 Numerical Methods

Tutorial : Numerical PDE & Boundary Value Problem

(Jan 2013)

1. Consider the following Poisson equation:

$$u_{xx} + u_{yy} = 4, \quad 0 < x < 1, 0 < y < 2;$$
$$u(x,0) = x^2, u(x,2) = (x-2)^2, 0 \le x \le 1;$$
$$u(0,y) = y^2, u(1,y) = (y-1)^2, 0 \le y \le 2;$$

   Use $h = k = 0.5$, approximate the solution to the Poisson equation by using:

   a) solving the matrix for $u_{i,j}$ directly.

   b) approximate $u_{i,j}$ using a Gauss-Seidel iterations for 5 times.

   Compare the results obtained with the actual solution $u(x,y) = (x-y)^2$.

2. A 6 cm-by-5 cm rectangular silver plate has heat being uniformly generated at each point at the rate $q = 1.5\text{cal/cm}^3\text{s}$. Let x represent the distance along the edge of the plate of length 6 cm and y be the distance along the edge of the plate of length 5 cm. Suppose the temperature u along the edges is kept at the following temperatures:

$$u(x,0) = x(6-x), u(x,5) = 0, 0 \le x \le 6,$$
$$u(0,y) = y(5-y), u(6,y) = 0, 0 \le y \le 5,$$

   where the origin lies at a corner of the plate with coordinates $(0,0)$ and the edges lie along the positive x- and y-axes. The steady-state temperature $u = u(x,y)$ satisfies the Poisson's equation:

$$u_{xx} + u_{yy} = -\frac{q}{K}, \quad 0 < x < 6, 0 < y < 5,$$

   where K, the thermal conductivity is 1.04cal/cmKs. Approximate the temperature $u(x,y)$ using the parameters $h = 0.4$ and $k = 1/3$.

3. Consider the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, 0 < x < 1, t > 0,$$

   subject to the conditions

$$u(0,t) = u(1,t) = 0, t > 0, \text{ and } u(x,0) = \sin \pi x, 0 \le x \le 1.$$

   Using an x step size of $h = 0.1$ and a t step size of $k = 0.001$, use the FTCS scheme to estimate $u(0.1, 0.002)$. Compare your answer to the exact solution $u(x,t) = \exp(-\pi^2 t) \sin \pi x$.

4. Resolve the heat equation in the previous question by using a Crank-Nicolson Scheme with an x step size of $h = 0.1$ and a t step size of $k = 0.01$. Estimate $u(0.1, 0.02)$.

5. Let $T_j^n \approx T(x_j, t_n)$. Explain how you could obtain the Forward-Time Centered-Space (FTCS) difference scheme

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = \frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{(\Delta x)^2}$$

for the heat equation $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$. Let $T_j^n = A^n \exp(ijk\Delta x)$ where k is an integer and $i = \sqrt{-1}$. Show that $A = 1 - \dfrac{4\Delta t}{(\Delta x)^2} \sin^2(\dfrac{k\Delta x}{2})$. Hence or otherwise find the *stability condition* for the scheme.

6. Repeat the previous question for the Backward-Time Centered-Space (BTCS) scheme. Show that $A = \dfrac{1}{1 + \frac{4\Delta t}{(\Delta x)^2} \sin^2(\frac{k\Delta x}{2})}$. Hence or otherwise deduce that the scheme is always stable.

7. Consider the following wave equation:

$$\begin{aligned} u_{tt} - u_{xx} = 0, \quad 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0, t > 0, \\ u(x,0) = \sin 2\pi x, 0 \le x \le 1, \\ u_t(x,0) = 2\pi \sin 2\pi x, 0 \le x \le 1. \end{aligned}$$

Use a x step size $h = 0.1$; time step size $k = 0.1$; and approximate the solution to the wave equation for $0 < t \le 0.3$. Compare the results obtained at $t = 0.3$ to the actual solution $u(x, y) = \sin 2\pi x(\cos 2\pi t + \sin 2\pi t)$.

8. The air pressure $p(x,t)$ in an organ pipe is governed by the wave equation $p_{tt} - c^2 p_{xx} = 0$, $0 < x < l, t > 0$, where l is the length of the pipe, and c is a physical constant. If the pipe is open, the boundary conditions are given by

$$p(0,t) = p_0, \quad \text{and} \quad p(l,t) = p_0.$$

If the pipe is closed, the boundary conditions are

$$p(0,t) = p_0, \quad \text{and} \quad p_x(l,t) = 0.$$

Assume that $c = 1, l = 1$, and the initial conditions are

$$p(x,0) = p_0 \cos 2\pi x, \quad \text{and} \quad p_t(x,0) = 0, 0 \le x \le 1.$$

a) Approximate the pressure for an open pipe with $p_0 = 0.9$ at $x = 0.5$ for $t = 0.5$ and $t = 1$, using step sizes $h = k = 0.1$.

b) Approximate the pressure for a closed pipe with $p_0 = 0.9$ at $x = 0.5$ for $t = 0.5$ and $t = 1$, using step sizes $h = k = 0.1$.