1.0 Attribute Statistical Process Control Chart ........................................ 3
1.1 Monitoring Proportion of Defective Part in a Lot ............................. 3
   Example 1 ........................................................................................................ 4
1.2 Monitoring Number of Defective Part in a Lot ................................. 5
   Example 2 ........................................................................................................ 5
1.3 Monitoring Number of Defective Part ............................................... 6
   Example 3 ........................................................................................................ 7
1.4 Monitoring Average Number of Defective Part ................................. 8
   Example 4 ........................................................................................................ 9
   Example 5 ....................................................................................................... 10
1.5 Average Run Length of $p$-Control Chart .......................................... 11
   Example 6 ....................................................................................................... 12
Figure 1: Data of example 1 ................................................................. 4
Figure 2: Attribute statistical process control chart for example 2 ............ 6
Figure 3: Data of example 3 ................................................................. 7
Figure 4: c control chart of example 3 .................................................. 8
Figure 5: Data of example 4 ................................................................. 9
Figure 6: U control chart of example 4 .................................................. 10
Figure 7: Data of example 5 ................................................................. 10
Figure 8: Control limits of U chart for $i = 1, 2, 3, 4, 5, 6, 7,$ and 8 .......... 11
Figure 9: U control chart of example 5 .................................................. 11
1.0 Attribute Statistical Process Control Chart

In the case where quality is measured as attribute such as the number of defect in a component or a product or a batch of components or products, number or proportion of defectives in a batch, etc., attribute control charts are used. The attribute control charts that will be discussed are proportion of defective part in a lot, the number of defective part in a lot, number of defective part, and average number defective part.

1.1 Monitoring Proportion of Defective Part in a Lot

The proportion of defective part in a lot is denoted by \( p \). It is defined as the number of defective part divided by the size in a batch. The test hypothesis is the null hypothesis \( H_0: p = p_0 \) versus the alternative \( H_1: p \neq p_0 \). Note that for a manufacturer, it wishes that \( p_0 \) is the target whereby it should be as small as possible.

The test statistic is the sample proportion of defectives \( \hat{p} \) or \( \hat{p} = x/n \), where \( n \) is the sample size of the batch and \( x \) is number of defective part in the sample batch. The general formulae of control limit are \( \sigma_0 = \text{Stdev}(\hat{p}) = \sqrt{\frac{p_0(1-p_0)}{n}} \). Hence the control limits are

\[
\text{LCL or UCL} = p_0 \pm Z_{\alpha/2} \sqrt{\frac{p_0(1-p_0)}{n}}
\]

For \( \alpha = 0.0026 \), the control limits are \( p_0 \pm 3 \sqrt{\frac{p_0(1-p_0)}{n}} \). The central line is \( p_0 \).

If \( p_0 \) value is not specified, it must be estimated from the data. Let’s assume that there are \( m \) sample batches, each with sample size \( n \) are collected and that the total number of defective part in these \( m \) sample batches is \( d \) then the estimate of the proportion of defectives per sample batch of sample size \( n \) is \( \bar{p} = d/(mn) \), \( \hat{p}_0 \) and \( \bar{p} \). Then the control limits are

\[
\text{LCL} = \bar{p} - 3 \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}
\]

\[
\text{CL} = \bar{p}
\]

\[
\text{UCL} = \bar{p} + 3 \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}
\]

- 3 -
The control chart is called a $p$ chart.

**Example 1**

The readings in Fig. 1 are the number of defective parts in 18 sample batches, each containing a total of 50 items i.e. $n = 50$. Establish the control limits.

**Solution**

The total number of defective parts in the 18 sample batches collected is 229. Each sample batch has 50 items. Thus, the average fraction of defective in these 18 sample batches is $\bar{p} = 229/(18 \times 50) = 0.254$. The limits using average fraction of defective as the estimate of $p_0$ are $\bar{p} \pm 3 \sqrt{\frac{\bar{p}(1 - \bar{p})}{n}} = 0.254 \pm 3 \sqrt{\frac{0.254(1 - 0.254)}{50}}$, which are 0.0693 and 0.439 and the center line is 0.254.

<table>
<thead>
<tr>
<th>Batch Number $i$</th>
<th>Number of Defect $x_i$</th>
<th>Proportion of Defect $\bar{p}_i = x_i / 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>0.18</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>0.20</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>0.22</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>0.26</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
<td>0.26</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>0.16</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>0.36</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>0.24</td>
</tr>
<tr>
<td>9</td>
<td>11</td>
<td>0.22</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>0.16</td>
</tr>
<tr>
<td>11</td>
<td>14</td>
<td>0.28</td>
</tr>
<tr>
<td>12</td>
<td>21</td>
<td>0.42</td>
</tr>
<tr>
<td>13</td>
<td>18</td>
<td>0.36</td>
</tr>
<tr>
<td>14</td>
<td>10</td>
<td>0.20</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
<td>0.16</td>
</tr>
<tr>
<td>16</td>
<td>18</td>
<td>0.38</td>
</tr>
<tr>
<td>17</td>
<td>19</td>
<td>0.38</td>
</tr>
<tr>
<td>18</td>
<td>8</td>
<td>0.16</td>
</tr>
<tr>
<td>$\bar{p}$</td>
<td></td>
<td>0.254</td>
</tr>
</tbody>
</table>

**Figure 1:** Data of example 1

In this example, all the $\bar{p}$ values are within the calculated control limits. Hence, these limits can be used for monitoring $p$. If one or more $\bar{p}$ value falls outside the limits, then these values have to be removed and a new $\bar{p}$ is calculated. This procedure has to be repeated until all the $\bar{p}$ values used in the estimation of $\bar{p}$ are within the control limits.
1.2 Monitoring Number of Defective Part in a Lot

The number of defective part in a lot is denoted by $np$. The hypotheses tested are; null hypothesis $H_0$: mean number of defectives $= np_0$ versus alternative hypothesis $H_1$: mean number of defectives $\neq np_0$.

The test statistic used is the number of defective part $x$ in a sample batch of size $n$, which is denoted as $np$. The control limits of this chart are obtained by multiplying the lower and upper control limits of the $p$ chart by $n$, which are

$$\text{LCL or UCL} = n \left[ p_0 \pm 3 \sqrt{\frac{p_0(1-p_0)}{n}} \right]$$

for $\alpha = 0.0026$ (5)

also are equal to $np_0 \pm 3\sqrt{np_0(1-p_0)}$.

In equation (5), $p_0$ can be replaced by its estimate, which is $\bar{p}$. The center line CL is $CL = np_0$. This is called $np$ chart.

Example 2
Using data shown in Fig. 1, establish the control limits for number of defective part in the lot and plot the control chart.

Solution
The control limits are calculated using $\bar{p} = 0.254$ and $n = 50$, and equation

$$\text{LCL or UCL} = n \left[ p_0 \pm 3 \sqrt{\frac{p_0(1-p_0)}{n}} \right]$$

which is $50 \left[ 0.254 \pm 3 \sqrt{\frac{0.254(1-0.254)}{50}} \right]$ after substituting $p_0$ value and $n$ value.

It yields lower control limit LCL equal to 3.45 and upper control limit UCL equal to 21.95.

The statistical control chart is shown in Fig. 2. It may sound ridiculous to have fractional control limits since the data is non parametric type. Nevertheless, it is acceptable for academic purpose.
1.3 Monitoring Number of Defective Part

If the quality of a component or product is measured in terms of the number of defect per component or product or batch then a \( c \) control chart is used. The letter \( c \) here denotes the number of defect per component or product or per some appropriate units of the product. Examples are the number of defect per 10 yards of a cable or per 1.0 square yard of an aluminum sheet, etc. or the number of defect in a sample of size \( n \).

The test hypothesis is; null hypothesis \( H_0 \): mean number of defect per piece or some units or per batch = \( c_0 \) versus alternative hypothesis \( H_1 \): mean number of defect per piece or some units or product \( \neq c_0 \).

The test statistic is the number of defect per the appropriate unit, which is \( c \). The expected value and the standard deviation with assumption that \( c \) obeys a Poisson distribution are \( E(c) = c_0 \); standard deviation of \( c = c_0 \). Poisson distribution are \( E(c) = c_0 \) and standard deviation of \( c = c_0 \). Hence, the control limits \( c_0 \pm Z_{\alpha/2} \sqrt{c_0} \) becomes equation (6) taking \( \alpha \) is \( \alpha = 0.0026 \).
\[ c_0 \pm 3\sqrt{c_0} \quad \text{(6)} \]

The center line is \( c_0 \). If \( c_0 \) cannot be specified, it can be estimated by the mean number of defects from one or more sample batches, which is denoted by \( \bar{c} \) then, the control limits become \( \bar{c} \pm 3\sqrt{c} \). The center line is located at \( \bar{c} \).

**Example 3**

Data shown in Fig. 3 are the number of defects in 10 sample batches, where each batch contains 15 items. Establish the control limits of the number of defects in each batch and plot the control chart.

<table>
<thead>
<tr>
<th>Batch number ( i )</th>
<th>Number of Defects in Batch ( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
</tr>
<tr>
<td>9</td>
<td>19</td>
</tr>
<tr>
<td>10</td>
<td>17</td>
</tr>
</tbody>
</table>

*Figure 3: Data of example 3*

**Solution**

Assuming that the process is in control when these observations are collected, the calculated limits of the \( c \) chart are \( \bar{c} = \frac{18 + 12 + 14 + 9 + 16 + 10 + 14 + 15 + 19 + 17}{10} = 12.4 \).

The control limits are LCL = 12.4 - 3\( \sqrt{12.4} \) = 1.84 and UCL = 12.4 + 3\( \sqrt{12.4} \) = 22.96. The center line CL is 12.4. All the \( c \) values used in the estimation are within the control limits as is shown in Fig. 4.
1.4 Monitoring Average Number of Defective Part

If the quality is measured in terms of the average number of defect per unit and it is not the actual number of defects per unit or the actual number of defect per sample batch then a U control chart is used. Here, $U$ denotes the average number of defects per unit and $c$ denotes the actual number of defect per sample batch containing $n$ items.

If $U_0$ denotes the in-control or targeted mean of the average number of defect per unit then the hypothesis being tested is; null hypothesis $H_0$: mean number of average defect per unit equal to $U_0$ versus alternative hypothesis $H_1$: mean number of average defect per unit is not equal $U_0$.

The test statistic used is the average number of defect per unit $U$. The expected value $E(U)$ and standard deviation of $U$ are

\[ E(U) = U_0 \]  
\[ U = \frac{c}{n} \]  
\[ \sqrt{\frac{U_0}{n}} \]

where $U = c/n$; $E(c) = c_0$; $\text{Var}(c) = c_0$; $E(U) = c_0/n = U_0$; $\text{Var}(U) = \frac{c_0}{n^2} = nU_0/n^2 = U_0/n$. $n$ is the sample size. Hence, the lower and upper control limits of the U chart are

\[ \text{LCL or UCL} = U_0 \pm Z_{\alpha/2} \sqrt{\frac{U_0}{n}} \]
or \( U_0 \pm 3\sqrt{U_0/n} \) for \( \alpha = 0.0026 \). The center line is at \( U_0 \). If \( U_0 \) cannot be specified, it is estimated by the sample mean number of defects per unit from observations collected when the process is in control. Let this estimate be \( \bar{U} \) then the control limits are

\[
\bar{U} \pm 3\sqrt{\bar{U}/n}
\]

The center line CL is \( \bar{U} \).

**Example 4**
The data set shown in Fig. 5 contains the number of defect in eight sample batches, where each batch contains 15 items. Assuming that the process is in control when these observations are collected, calculate the control limits of the \( U \) statistical process control chart and plot the control chart.

<table>
<thead>
<tr>
<th>Batch Number ( i )</th>
<th>Number of Defect in Batch ( c )</th>
<th>Average Number of Defect per unit ( \bar{U} = c/15 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18</td>
<td>1.20</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>0.80</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>0.47</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>0.60</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>1.07</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>0.80</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>0.93</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
<td>1.00</td>
</tr>
</tbody>
</table>

\[
\bar{U} = \frac{\sum_{i=1}^{8} c_i}{8 \times 15} = 0.86
\]

*Figure 5: Data of example 4*

**Solution**
The average number of defect \( \bar{U} \) is \( \bar{U} = [18 + 12 + 7 + 9 + 16 + 12 + 14 + 15]/(8 \times 15) = 0.86 \).

The control limits are \( \bar{U} \pm 3\sqrt{\bar{U}/n} = 0.86 \pm 3\sqrt{0.86/15} \), which are 0.14 and 1.58. The center line CL is 0.86. The statistical control chart is shown in Fig. 6.
Example 5

The data of the number of defect found in eight sample batches with unequal sample batch size are given in Fig. 7. Calculate the control limits of the U chart and plot the control chart.

<table>
<thead>
<tr>
<th>Batch Number $i$</th>
<th>Sample Size $n_i$</th>
<th>Number of Defect in Batch $c_i$</th>
<th>Average Number of Defect per unit $U_i = c_i/n_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>5</td>
<td>$5/2 = 2.5$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
<td>$6/3 = 2.0$</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>$2/2 = 1.0$</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>7</td>
<td>$7/5 = 1.4$</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>9</td>
<td>$9/8 = 1.1$</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>15</td>
<td>$15/10 = 1.5$</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>7</td>
<td>$7/3 = 2.3$</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>14</td>
<td>$14/7 = 2.0$</td>
</tr>
<tr>
<td>Total</td>
<td>40</td>
<td>65</td>
<td>$\bar{U} = 65/40 = 1.63$</td>
</tr>
</tbody>
</table>

Solution

In this example, the sample sizes are not equal. Hence each sample batch will have its own control limits. This is because the control limits $U \pm 3\sqrt{U/n}$ are functions of the sample size. Grand mean $\bar{U}$ is calculated using all observations. Thus, $\bar{U}$ is $\bar{U} = [5 + 6 + 2 + 7 + 9 + 15 + 7 + 14]/[2 + 3 + 2 + 5 + 8 + 3 + 7] = 65/40 = 1.63$.

The control limits are equal to $\bar{U} \pm 3\sqrt{\bar{U}/n_i}$, where the subscript $i$ denotes batch $i$. The center line CL is 1.63.

For $i = 1$, the control limits are $1.63 \pm 3\sqrt{1.63/2}$, which are -1.08 and 4.34 respectively. Since negative value is not permitted, the control limits for $i = 1$ are lower control
Soo King Lim

The control limits of eight sample batches are given in Fig. 8. It can be seen that the average numbers of defect $U_i$’s for all sample batches are within their respective control limits. The U control chart is shown in Fig. 9.

\begin{figure}[h]
\centering
\includegraphics[width=\columnwidth]{chart.png}
\caption{U control chart of example 5}
\end{figure}

1.5 Average Run Length of $p$-Control Chart

The proportion of defective part in the lot is $\bar{p}$ or $\hat{p} = x/n$, where $n$ is the sample size of the batch and $x$ is number of defective part in the sample batch. The formulae of control limits are $\bar{p} \pm Z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}}$. For $\alpha = 0.0026$, they are equal to
\[ p \pm 3\sqrt{\frac{p(1-p)}{n}}. \] The probability \( q \) for \( m \) out of \( n \) samples from the lot is defective is given by

\[ q = \left( \frac{p}{n} \right)^m \tag{11} \]

Thus, the average run length of \( p \)-control chart is

\[ \text{ARL} = \frac{1}{\left( \frac{p}{n} \right)^m} \tag{12} \]

**Example 6**

Let’s take the earlier illustration of Shewart control chart that sample size is \( n = 6 \), the probability of the plotted point outside the control limit is \( p = 1 - \Pr[-5.44 < Z < 0.55] = 1 - (0.7088 - 0) = 0.2912 \) and the ARL is 3.43. Calculate the ARL of \( p \)-control chart using this data for 2 out of 6 and 6 out of 6 samples are defective. Comment your results.

**Solution**

The average run length of the \( p \)-control chart for 2 out of 6 samples are defective is

\[ \text{ARL} = \frac{1}{\left( \frac{p}{n} \right)^m} = \frac{1}{(0.2912)^2} = 11.79. \]

The average run length of the \( p \)-control chart for 6 out of 6 samples are defective is

\[ \text{ARL} = \frac{1}{\left( \frac{p}{n} \right)^m} = \frac{1}{(0.2912)^6} = 1640. \]

For 2 out of 6 samples are defective case, it would most probable occur once between 11 and 12 sample batches.

For 6 out of 6 samples are defective case, it would most probable occur once every 1,640 sample batches.
A
Average run length ........................................ 10
C
c control chart ........................................... 5
N
np chart ..................................................... 4

P
p chart ..................................................... 3
U
U control chart ........................................... 7