Chapter 9

Magnetism

9.1 Introduction

Permanent magnets have long been used in navigational compasses. Magnet always has two poles. Unlike the case where electric charge can be separated into positive and negative charge, so far no one has found a magnetic monopole. Any attempt to separate the north and south poles by cutting the magnet fails because each piece becomes a smaller magnet with its own north and south poles.

In this chapter, we cover the basic fundamentals of magnetism that includes magnetic force and electromagnetism, and induction.

9.2 Magnetic Field

Magnetic fields are produced by electric currents, which can be macroscopic currents in wires or microscopic currents associated with electrons in atomic orbits. Magnetic field sources are essentially dipolar in nature, having a north and south magnetic pole. The SI unit for magnetic field is Tesla, which can be seen from the magnetic part of the Lorentz force law \( \vec{F} = q \vec{V} \times \vec{B} \) to be composed of (Newton x second)/(Coulomb x meter). A smaller magnetic field unit is the Gauss (1 Tesla = 10,000 Gauss). Note that Lorentz force law is \( \vec{F} = q \vec{E} + q \vec{V} \times \vec{B} \).

There are many source of magnetic field. Some illustrations are shown in Fig. 9.1.

![Figure 9.1: Sources of Magnetic field](image-url)
9.2.1 Magnetic field of the earth

The earth's magnetic field is similar to that of a bar magnet tilted 11 degrees from the spin axis of the earth. The Curie temperature of iron is about 770 °C, which lower than the earth's core temperature, whereby it should not have magnetic field. However, magnetic fields surround electric currents that surmise circulating electric currents in the Earth's molten metallic core gives rise to magnetic field.

The earth's magnetic field is attributed to a dynamo effect of circulating electric current, but it is not constant in direction. Rock specimens of different age in similar locations have different directions of permanent magnetization. Evidence for 171 magnetic field reversals during the past 71 million years has been reported.

Although the details of the dynamo effect are not known in detail, the rotation of the Earth plays a part in generating the currents, which are presumed to be the source of the magnetic field. Venus does not have such a magnetic field although its core has iron content similar to that of the Earth. But the Venus's rotation period is 243 Earth days that is too slow to produce the dynamo effect.

9.2.2 Magnetic flux

Magnetic flux $\Phi_B$ is the product of the average magnetic field times the perpendicular area that it penetrates i.e. $\Phi_B = B \cdot A \cdot$. It is also equal to $BA \cos \theta$, where $\theta$ is the angle between the magnetic field and the plane of the area. The illustration is shown in Fig. 9.2.

![Illustration of magnetic flux](image)

**Figure 9.2**: Illustration of magnetic flux
For a closed surface, the sum of magnetic flux $\Phi_B$ is always equal to zero, which is Gauss' law for magnetism i.e.

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$  \hspace{1cm} (9.1)

No matter how small the volume, the magnetic sources are always dipole sources so that there are as many magnetic field lines coming in (to the south pole) as out (from the north pole).

### 9.2.3 Magnetic field of a current loop

Electric current in a circular loop creates a magnetic field as shown in Fig. 9.3, which is more concentrated in the center of the loop than outside the loop. Stacking multiple loops called a solenoid concentrates more fields.

**Figure 9.3:** Magnetic field of a current loop

### 9.2.4 Magnetic field contribution of a current element

The Biot-Savart Law relates magnetic fields to the currents, which are their sources. In a similar manner, Coulomb's law relates electric fields to the point charges, which are their sources. Biot-Savart’s law states that the magnetic field of a current element $d\mathbf{B} = \frac{\mu_0 I dl \times \hat{r}}{4\pi r^2}$, where $\mu_0$ is equal to $4\pi \times 10^{-7}$ TmA$^{-1}$. The illustration is shown in Fig. 9.4.
Magnetic field at the center of current loop
The magnetic field at the center of current loop can be derived from Biot-Savart’s law, which states $d\vec{B} = \frac{\mu_0 I d\vec{L} \times \hat{r}}{4\pi r^2}$ and the illustration shown in Fig. 9.5.

From Biot-Savart’s law, $d\vec{B} = \frac{\mu_0 I d\vec{L} \times \hat{r}}{4\pi r^2} = dB = \frac{\mu_0 I dL \sin \theta}{4\pi R^2}$. Since the angle between $d\vec{L}$ and $\hat{r}$ is 90°, therefore $\sin \theta = \sin 90^\circ = 1$. Biot-Savart’s equation shall be $B = \frac{\mu_0 I}{4\pi R^2} \int dL$. Therefore the magnetic field at the center of current loop is

$$\vec{B} = \frac{\mu_0 I}{2R} \hat{r} \quad (9.2)$$

Magnetic field on the axis of current loop
The magnetic field at the axis of current loop can be derived from Biot-Savart’s law, which states $d\vec{B} = \frac{\mu_0 I d\vec{L} \times \hat{r}}{4\pi r^2}$ and the illustration shown in Fig. 9.6.
The magnetic field at $x$-direction cancels each other due to symmetrical direction. The only component left is the $z$-component.

The $z$-component magnetic field $\mathbf{d\bar{B}}_z = \frac{\mu_0 I dL x \hat{r}}{4\pi r^2} \cos \theta$. $r$ is equal to $\sqrt{R^2 + z^2}$.

Therefore, the $z$-component magnetic field is $\mathbf{d\bar{B}}_z = \frac{\mu_0 I dL R \hat{k}}{4\pi(z^2 + R^2)^{3/2}}$. After integrating $\int dL = 2\pi R$, the $z$-component magnetic field

$$\mathbf{\bar{B}}_z = \frac{2\pi \mu_0 I R^2}{4\pi(z^2 + R^2)^{3/2}} \hat{k} \quad (9.3)$$

### 9.2.5 Ampere’s Law

The net electric field due to any distribution of charges follows equation (8.3) i.e. $\mathbf{\bar{E}} = \left(\frac{q}{4\pi \varepsilon_0 r^2}\right) \hat{r}$. Similarly the net magnetic field due to any distribution of current follows Biot-Savart’s law $\mathbf{d\bar{B}} = \frac{\mu_0 I dL x \hat{r}}{4\pi r^2}$. If the distribution has some symmetry, then Ampere’s law can be used to find the magnetic field with less effort. Ampere’s law states that the sum of the magnetic field along any closed path is proportional to current that passes through. Mathematically, it is

$$\oint \mathbf{\bar{B}} \cdot d\mathbf{s} = \oint \mathbf{B} \cos \theta ds = \mu_0 i_{\text{enc}} \quad (9.4)$$

Figure 9.6: Magnetic field at the axis of current loop
where \( ds \) is the length of Amperian loop and \( B \cos \theta \) is tangential to the loop as shown in Fig. 9.7.

**Figure 9.7:** Ampere’s law applied to an arbitrary Amperian loop encircles two long wires and excluding a third wire. The ‘dot’ indicates the current is out of the page and “cross” indicates the current is into the page

Applying ampere’s law for the above Amperian loop, it becomes

\[
\oint B \cdot ds = \oint B \cos \theta ds = \mu_0 (i_1 - i_2).
\]

**Magnetic field outside a long straight wire with current**

Consider the case where a long wire carries current \( i \) directly out of the page as shown in Fig. 9.8. The magnitude of magnetic field \( B \) at distance \( r \) from the wire has a cylindrical symmetry. Thus, applying Ampere’s law, the magnetic field at the distance \( r \) around the wire is \( \oint B ds \cos \theta = \mu_0 i \). Knowing that \( \theta = 0 \), \( i_{enc} = i \), and \( \oint ds = 2\pi r \), the magnetic shall be

\[
B = \frac{\mu_0 i}{2\pi r}
\]  

(9.5)

**Figure 9.8:** Magnetic field due to long wire carrying current \( i \)
Magnetic field inside a long straight wire with current
Consider the case for the magnetic field inside a long straight wire with current shown in Fig. 9.10.

![Magnetic field inside a long wire with current](image)

**Figure 9.10:** Magnetic field inside a long wire with current

Applying Ampere’s law $\oint B \cos \theta = \mu_0 i_{enc}$, where $i_{enc}$ is equal to $i_{enc} = \frac{\pi R^2}{2\pi R^2} i$, the magnetic field $B$ is equal to

$$B = \left( \frac{\mu_0 i}{2\pi R^2} \right) r \quad (9.6)$$

Magnetic field of solenoid
A tight wound helical coil of wire is called solenoid as shown in Fig. 9.11(a). The magnetic field at the center of the solenoid that carrying current $i$ and has $n$ turn per unit length can be calculated using Ampere’s law.

![Solenoid and Amperian loop of the solenoid](image)

**Figure 9.11:** (a) Solenoid and (b) Amperian loop of the solenoid
Consider the Amperian rectangular loop $abcd$ as shown in Fig. 9.12(b). Apply Ampere’s law, \[ \oint_{\text{enc}} B \cdot d\mathbf{s} = \mu_0 N i \] to the Amperian loop, it is \[ \int_B \cdot d\mathbf{s} = \int_b^c B \cdot d\mathbf{s} + \int_c^d B \cdot d\mathbf{s} + \int_d^a B \cdot d\mathbf{s} = \mu_0 N i, \] where $N$ is the number of turn in the Amperian loop. \[ \int_b^c B \cdot d\mathbf{s} = \int_c^d B \cdot d\mathbf{s} = \int_d^a B \cdot d\mathbf{s} = 0 \] since $ad$ and $bc$ are perpendicular to the magnetic field and outside the solenoid the magnetic field is assumed to be zero. Thus, the magnetic field at the center of solenoid is

\[ B = \frac{\mu_0 Ni}{h} = \mu_0 ni \]  

(9.7)

where $N/h = n$, the number of turn per unit length.

**Magnetic field of toroid**

A toroid can be considered as a solenoid bent into a shape of hollow doughnut as shown in Fig. 9.12(a). The magnetic field at the center of the toroid that carrying current $i$ and has $N$ turn can be calculated using Ampere’s law.

\begin{align*}
B &= \frac{\mu_0 Ni}{h} = \mu_0 ni
\end{align*}

(9.8)
9.3 Magnetic Force Exerts On Moving Charge

In previous chapter, the electric field is determined by placing a positive test charge \( q_0 \) at rest, measuring the electric force acting on the particle, and calculating the electric field based on equation \( E = \frac{F}{q_0} \). If a magnetic monopole were available, the magnetic field \( B \) can be determined in a similar manner. Thus, the magnetic field \( B \) is determined from a different manner in terms of magnetic force \( \vec{F}_B \) exerted on the moving test particle of certain velocity \( \vec{V} \). Therefore, magnetic force \( \vec{F}_B \) is defined as

\[
\vec{F}_B = q\vec{V} \times \vec{B}
\]  

(9.9)

Based on equation (9.9), the position of magnetic force exerted by a moving charge particle in the magnetic field \( \vec{B} \) can be determined by the right hand as shown in Fig. 9.13.

From the results shown in Fig. 9.13, the magnetic force \( \vec{F}_B \) acting on a charged particle moving with velocity \( \vec{V} \) through magnetic field \( \vec{B} \) is always perpendicular to velocity \( \vec{V} \) and magnetic field \( \vec{B} \).

\[\text{Figure 9.13: Right hand rule used to determine the direction of magnetic force}\]

From equation (9.9), the unit for magnetic field \( \vec{B} \) is \( \text{Newton} / (\text{Coulomb}) \text{meter} / \text{sec} \), which is defined as 1 tesla. In non-MKS system, 1 tesla is equal to \( 1 \times 10^4 \) gauss.

Equation (9.9) can also be written as

\[
\vec{F}_B = l\vec{V} \times \vec{B}
\]  

(9.10)

where \( V = l/t \) and \( I = q/t \). \( V \) is the velocity and \( l \) is the length of conductor.
Example 1
A 0.1 T magnet has a field pointing upward. The pole faces has 2.00 cm diameter. Find the force on 0.50 A current flowing eastward.

Solution
Equation (9.1) $\vec{F}_B = q\vec{V} \times \vec{B}$ can be rewritten as $\vec{F}_B = \vec{I} \times \vec{B}$. Thus, the force is $2.00 \times 10^{-2} \times 5.00 \times 0.1 = 1.0 \times 10^{-2}$ N. The direction of the force is out of the page.

Example 2
A uniform magnetic field with magnitude 1.2 mT is directed vertically upward throughout the volume of laboratory chamber. A proton with kinetic energy 5.3 MeV enters the chamber, moving horizontally from south to north. What magnetic deflecting force acts on the proton as it enters the chamber? What is the acceleration of the proton? The proton mass is $1.67 \times 10^{-27}$ kg and neglecting the Earth’s magnetic field.

Solution
The velocity of proton is $V = \sqrt{\frac{2 \times 5.3 \times 10^6 \times 1.602 \times 10^{-19}}{1.67 \times 10^{-27}}} = 3.2 \times 10^7$ m/s. The magnetic force is $F_B = qVB \sin \theta = 1.602 \times 10^{-19} \times 3.2 \times 10^7 \times 1.2 \times 10^{-3} \times \sin 90^0 = 6.1 \times 10^{-15}$ N.
The acceleration of proton is $6.1 \times 10^{-15} / 1.67 \times 10^{-27} = 3.7 \times 10^{12}$ m/s$^2$.

9.4 Magnetic Force between Two Conductors

The magnetic force between two wires carrying current $I_1$ and $I_2$ can be calculated based on the Fig. 9.14.

![Figure 9.14: Magnetic force between two wires carrying current](image)
According to Ampere’s law \( \oint B \cdot ds = \mu_0 i + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int E \cdot dA \), the magnetic field \( B \) generated by a conductor carrying current \( I_1 \) is equal to \( B = \frac{\mu_0 I_1}{2\pi r} \). The force exerted by the magnetic field \( B \) on conductor 2 shall be \( \vec{F}_{21} = I_1 x \vec{B} = \frac{\mu_0 I_1 I_2 l}{2\pi r} \). Similarly, the force exerted by magnetic field from conductor 2 on conductor 1 shall be \( \vec{F}_{12} = I_1 x \vec{B} = \frac{\mu_0 I_1 I_2 l}{2\pi r} \). Thus, in general, the force \( F_m \) between conductors is

\[
F_m = \frac{\mu_0 I_1 I_2 l}{2\pi r}
\]  

(9.11)

### 9.5 Magnetic Force on a Current Loop

The magnetic force \( F_B \) on a current loop as shown in Fig. 9.15, follows equation \( \vec{F}_B = -lI x \vec{B} \), where \( l \) is the length of wire and \( I \) is the current flow in the wire loop.

![Figure 9.15: Magnetic force on current loop](image)

### 9.6 Torque on a Current Loop

A current carrying experiences a force when it is placed in magnetic field. Similarly if a wire loop is suspended properly in magnetic field, the magnetic force produces a torque tends to rotate the loop. The illustration is shown in Fig. 9.16.
The force by both sides is \( \vec{F} = II \times \vec{B} \). The torque is equal to \( II B \frac{W}{2} \sin \theta \). The net torque \( \tau = II w \sin \theta = IAB \sin \theta \). If there are \( N \) current loop, then the net torque \( \tau \) is

\[
\tau = NIA \sin \theta \tag{9.12}
\]

NIA is also defined as magnetic moment \( \mu \), which is also defined as \( \vec{\mu} = NI \cdot \vec{A} \) in vector form as shown in Fig. 9.17.

Based on magnetic moment, the net torque is \( \vec{\tau} = \vec{\mu} \times \vec{B} \). Since this torque acts perpendicular to the magnetic moment, then it can cause the magnetic moment to precess around the magnetic field at the Larmor frequency.

The energy \( W \) necessary to overcome the magnetic torque and rotate the current loop from \( 0^\circ \) to \( 180^\circ \) is the integration of \(-\tau d\theta\) from \( 0^\circ \) to \( 180^\circ \) i.e \( W = -\int_0^{180^\circ} \tau d\theta = 2\mu B \).
The position where the magnetic moment is opposite to the magnetic field is said to have a higher magnetic potential energy as shown Fig. 9.18

![Diagram of magnetic moments](image)

**Figure 9.18:** Low and high energy magnetic moment

### 9.7 Induced Voltage

Moving a wire through the magnetic field causes the charges within the wire to feel an upward magnetic force \( F = q\vec{V} \times \vec{B} \) as shown in Fig. 9.19. Positive charges accumulate at the top of the wire and negative charges at the bottom. This creates a downward electric field in the wire. The net force on the charges is given by the Newton’s second law, which is \( ma = qVB - qE \). Charges move upward until acceleration is equal to zero i.e. \( a = 0 \). The electric field is then equal to \( E = VB \).

![Diagram of a conductor in a magnetic field](image)

**Figure 9.19:** Moving a conductor in magnetic field

If the wire was rolling along the rails connected to a voltmeter as shown in Fig. 9.20, the meter would give a reading due to the electric field in the wire. This voltage can be found from the electric potential equation \( \varepsilon = -\int\vec{E} \cdot d\vec{s} = -EI = -VBI \).
If one sets $V = \frac{dx}{dt}$, then $V_p$ is equal to induced voltage i.e. $\varepsilon = - Bl \frac{dx}{dt} = - \frac{d\Phi_B}{dt}$, where $ldx$ is area $A$ and $BA$ is the magnetic flux $\Phi_B$. Thus, one can see that if there is change of magnetic flux $\Phi_B$, an induced voltage is created. This equation $\varepsilon = - \frac{d\Phi_B}{dt}$ is for equation for the law of Faraday of induction. Thus,

$$\varepsilon = - \frac{d\Phi_B}{dt} \quad (9.13)$$

The negative sign indicating that the induced voltage is opposing the flux change.

### 9.8 Lenz’s Law

Soon after Faraday proposed his law of induction, Heinrich Friedrich Lenz devised a rule known as Lenz’s law for determining the direction of an induced current loop. Lenz states that an induced current has a direction such that the magnetic field due to the current opposes the change in the magnetic flux that induces the current. The illustration is shown in Fig. 9.21.
Consider the cases shown in Fig. 9.22. Case (a) and (c) show the increase of magnetic field $\vec{B}$ toward the loop, the induced magnetic field $\vec{B}_i$ is opposing the increase.

Case (b) and (d) show the decrease of magnetic field $\vec{B}$ away from the loop, the induced magnetic field $\vec{B}_i$ is opposing the decrease.

**Example 3**
Estimate the induced voltage across the 40.0 m wingspread of an airplane traveling at 222 m/s perpendicular to the earth’s magnetic field of 50.0 $\mu$T.

**Solution**
Using the second law when the charges in the wing have reached equilibrium when $qV\vec{B} = q\vec{E}$, which is $\vec{E} = V\vec{B}$. The induced voltage can be found from $\varepsilon = -\int \vec{E} \cdot d\vec{s} = -\varepsilon l = -VBl = 222 \times 40 \times 50 \times 10^{-6}$ T = 0.444 V.

**Example 4**
A small loop of $N$ turns and area $A$ is in the same plane as a long straight wire carrying a current $i = i_0 \sin \omega t$. Find the induced voltage in the loop as a function of time and show that the peak-to-peak voltage is proportional to the peak magnetic field.

**Solution**
The field due to a long straight wire is
B = \frac{\mu_0 i}{2\pi r}, \text{ where } r \text{ is the distance from the wire to the center of the loop. The loop is small enough that at any given time the field is approximately constant.}

Therefore, the magnetic flux through the small loop is \( \Phi_B = \frac{\mu_0 i}{2\pi r} \cdot \text{A} \).

Applying Faraday's law of induction, the induced voltage is \( \varepsilon = -\frac{d\Phi_B}{dt} \)

\[ = -\frac{NA\mu_0}{2\pi r} \cdot \frac{d}{dt}(i_0 \sin \omega t) = \frac{NA\mu_0 i_0 \omega}{2\pi r} \cdot \cos \omega t. \]

The amplitude of induced voltage is \( \frac{NA\mu_0 i_0 \omega}{2\pi r} \) i.e. when \( \cos \omega t = 1 \). The peak-to-peak induced voltage is \( \frac{2NA\omega\mu_0 i_0}{2\pi r} \).

The magnetic field is \( B = \frac{\mu_0 i_0 \sin \omega t}{2\pi r} \). Thus, the peak magnetic field \( B_0 \) is \( B_0 = \frac{\mu_0 i_0}{2\pi r} \). The peak-to-peak induced voltage is equal to \( \frac{2NA\omega\mu_0 i_0}{2\pi r} \), which is also equal to \( 2NAB_0\omega \). Thus, it is proportional to peak magnetic field for constant frequency.

### 9.9 Motor and Generator

A motor and a generator are physically the same device. A motor converts electrical energy into mechanical energy, while a generator converts mechanical energy into electrical energy.

#### 9.9.1 Motor

A motor is constructed from the loop of current carrying wire in a magnetic field as shown in Fig. 9.23. It turns because the loop that has a magnetic dipole moment \( \mu \) felt the torque in the direction of the magnetic field \( B \). The dipole gains momentum as it rotates trying to align with the field. After it crosses the field, the current in it is reversed, as is its dipole moment again tries to align with the field. This continual swapping of the current direction causes a continuous rotation of the coil converting electrical energy into the mechanical energy of rotation.
Figure 9.23: The basic construction of motor (compare with Fig. 9.16)

9.9.2 Generator

A generator is just a motor running backwards. Mechanical energy is used to turn the coil. For instance, this energy can come from water falling over a dam. The changing magnetic flux induces a voltage and therefore, a current in the coil according to Faraday's Law. The generator converts the mechanical energy into electrical energy.

Example 5

A coil of $N$ turns and area $A$ is rotated at a frequency $f$ about an axis perpendicular to a magnetic field $B$. Find (a) the induced voltage as a function of time, (b) the peak voltage and (c) the $rms$ voltage.

Solution

(a) When the coil makes an angle $\theta$ with the magnetic field the flux through the coils is $\Phi_B = NBA \cos \theta = NBA \cos \omega t$.

The induced voltage is $\varepsilon = -N \frac{d}{dt}(BA \cos \omega t) = NBA \omega \sin \omega t$.

The peak induced voltage $\varepsilon_{peak}$ is $\varepsilon_{peak} = NBA \omega$.

The $rms$ voltage value $\varepsilon_{rms}$ is $\varepsilon_{rms} = \frac{NBA \omega}{\sqrt{2}}$. Alternately the $rms$ voltage value can be calculated from the definition, which is $\varepsilon_{rms} = \sqrt{\varepsilon^2}$, where

$$\frac{1}{T} \int_0^T \varepsilon^2 dt = \frac{\varepsilon_{peak}^2}{2}.$$
Example 6
A coil of 1,000 turns and 12.0 cm radius flips 180° about an axis that points northward. The coil has a resistance of 4.80 Ω. The vertical component of Earth's magnetic field is 46.0 µT. Find the total charge that flows when the coils flips.

Solution
Notice that the horizontal component of Earth's field contributes no flux. All the flux through the loop is due to the vertical component.

The induced voltage is \( \varepsilon = -N \frac{d\Phi_B}{dt} \).

According to Ohm's law the induced is also equal to \( \varepsilon = R \frac{dq}{dt} \). This implies that \( R \frac{dq}{dt} = -N \frac{d\Phi_B}{dt} \) or \( \int R dq = -N \int d\Phi_B \), also equal to \( RQ = -N(\Phi_B - \Phi_{B0}) \).

The initial flux \( \Phi_{B0} \) is just the product of the vertical component of the field and the area. The final flux is just the opposite of the initial flux. Thus, \( RQ = -N(\Phi_B - \Phi_{B0}) = -N(-B_V \pi r^2 - B_V \pi r^2) = 2B_V \pi r^2 \).

The total charge \( Q \) that flows when the coils flipped is

\[
Q = \frac{2NB_V \pi r^2}{R} = \frac{2 \times 1000 \times 46 \times 10^{-6} \pi (0.12)^2}{4.8} = 8.67 \times 10^{-4} \text{ C.}
\]

9.10 Self and Mutual Inductance

If there is current \( i \) in windings of solenoid or inductor, the current produces a magnetic flux \( \Phi_B \) through the central region of the inductor. The inductance of the inductor is then defined as

\[
L = \frac{N\Phi_B}{i} \quad (9.14)
\]

Since the magnetic flux \( N\Phi_B \) is equal to \( nl(BA) \) and also from equation (9.7) where \( B = \frac{\mu_0 Ni}{h} = \mu_n ni \), the inductance \( L \) is also equal to \( L = \mu_0 n^2 A \).
If the current in a coil is changed by varying the contact position on a variable resistor or applying an ac voltage as shown in Fig. 9.24, a self-induced voltage will appear in the coil while the current is changing. This process is called *self-induction*.

![Circuit Diagram](image)

*Figure 9.24: Circuit shows varying current to produce induced voltage*

Combining Faraday’s equation and equation (9.14), the self-induced voltage is equal to

\[ \varepsilon_L = -L \frac{di}{dt} \]  

(9.15)

If coil 1 carrying a current \(i_1\) is near coil 2 with \(N_2\) turns as shown in Fig. 9.25, the magnetic field caused by coil 1 will create a flux through the coil 2. If the current in the coil 1 changes, the magnetic flux through coil 2 will change. And according to Faraday's law, a voltage will be induced in the second 2. If the current in coil 2 changes, an induced voltage will be on coil 1. The relationship between the voltages induced in coil 2 and the rate of change of current in the coil 1 and vice versa is called *mutual inductance*. 
Figure 9.25: Illustration of mutual induction. (a) If the current in coil 1 changes, an induced voltage is on coil 2. (b) If the current in coil 2 changes, an induced voltage is on coil 1.

To find voltage induced in coil 2, one can use Faraday's Law that is

\[ \varepsilon_2 = -N_2 \frac{d\Phi_{21}}{dt} \]

The magnetic flux through coil 2 caused by coil 1 is \( \Phi_{21} = \int_{\text{coil2}} \vec{B}_1 \cdot d\vec{A}_2 \). The magnetic field caused by coil 1 can be found by the Biot-Savart's law, which is \( \vec{B}_1 = \frac{\mu_0 i_1}{4\pi} \int \frac{d\vec{s} \times \hat{r}}{r^2} \). The magnetic flux \( \Phi_{21} \) shall be

\[ \Phi_{21} = \frac{\mu_0 i_1}{4\pi} \int \int \frac{d\vec{s} \times \hat{r}}{r^2} \cdot d\vec{A}_2 = \frac{i_1}{N_2} M_{21}, \text{ where } M_{21} = \frac{\mu_0 N_2}{4\pi} \int \int \frac{d\vec{s} \times \hat{r}}{r^2} \cdot d\vec{A}_2. \]

Notice that \( M_{21} \) only depends on the shapes of the coils and the relative positions not on the current in coil 1 or on time.

Putting the magnetic flux into Faraday's law, the induced voltage on coil 2 is

\[ \varepsilon_2 = -N_2 \frac{d}{dt} \left( \frac{i_1}{N_2} M_{21} \right) = -M_{21} \frac{di_1}{dt} \]

(9.16)
Similarly, the induced voltage on coil 1 is

\[ \varepsilon_i = -N_1 \frac{d}{dt} \left( \frac{i_1}{N_1} M_{12} \right) = -M_{12} \frac{di_1}{dt} \]  

(9.17)

Equation (9.16) and (9.17), one can see the induced voltage is proportional to the rate of change of current in other coil. The constant of proportionality of equation (9.16) and (9.17) is indeed can be shown to be equal. i.e. \( M_{12} = M_{21} = M \). The SI unit for M is henry which is same inductance.

**Tutorials**

1. A straight horizontal length of copper wire has a current \( I = 28 \text{ A} \) through it. What is the magnitude and direction of the minimum magnetic field \( B \) needed to suspend the wire i.e. to balance the gravitational force on it? Given that the linear density of the wire is 46.6 g/m.

2. In the mass spectrometer, initially an ion of mass \( m \) and charge \( q \) is produced from a source S. The ion is accelerated by an electric field due to potential difference \( V \). The ion enters a separator chamber in which a uniform magnetic field \( B \) is perpendicular to the path of the ion. The magnetic field course the ion to move in a semicircle striking a photographic plate at distance \( x \) from the enter slit. If \( B = 0.08 \text{ mT} \), \( V = 1.000.0 \text{ V} \), and \( x = 1.6254 \text{ m} \), \( q = 1.602 \times 10^{-19} \text{ C} \), calculate the mass \( m \) of the ion in unified atomic mass units (1u = 1.6605x10^{-27} \text{ kg}).
3. Analog voltmeter and ammeter work by measuring the torque exerted by a magnetic field on current-carrying coil. A galvanometer is where both the design for voltmeter and ammeter based on. If the coil of a galvanometer is 21 cm high, 1.2 cm wide and has 250 turns. It is seated in the uniform radial magnetic field of 0.23 T. A spring provides counter torque to balance the magnetic torque given by the current flow in the coil. This results in a steady angular deflection $\phi$. If the current of 100 $\mu$A produces an angular deflection of 28° what must be the torsional constant $k$ of the spring, where its torque $\tau = -k\phi$?

4. Two circular close-packed coils are shown in the figure. The smaller one has radius $R_2$ with $N_2$ turns being coaxial with the larger one of radius $R_1$ with $N_1$ turn in the same plane. Derive an expression for the mutual inductance $M$ with assumption that $R_1 >> R_2$. What is the value of $M$ for $N_1 = N_2 = 1,200$ turns, $R_2 = 1.1$ cm and $R_1 = 15$ cm? Note that $\mu_0 = 4\pi \times 10^{-7}$ Tm/A.