Measurement of Uncertainty

Course

for

Engineer

Lim Soo King
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Chapter 1

Introduction

1.1 Aim of the Course

This course has been developed mainly for use by engineer and technician working in calibration laboratory, semiconductor assembly and test facility that perform calibration of inspection equipment, measuring equipment and test equipment. The course is also suitable for test/process engineer and maintenance that need to determine the correlation of test measurement parameter of product, process correlation of measurement, and error of measurement equipment. Under ISO9000 series of standards, QS 9000 and Laboratory Accreditation Standard (IOS/IEC Guide 25 or ISO/IEC 17025), it states that measurement uncertainty of the equipment or instrument used for inspection, measurement, and test is required to be known and specified for the user of the equipment/instrument.

This course has the aim to teach the user the basis and important aspects of measurement uncertainty, its application, the method of calculation, and expression of uncertainty measurement which is required by ISO "Guide to the Expression of Uncertainty in Measurement also called ISO/TAG4. The concept of NIS3003 Standard in terms of "The Expression of Uncertainty and Confidence in Measurement for Calibration" is also discussed.

1.2 Origin of ISO/TAG4

In 1978, the comite International des Poids et Measures (CPIM) recognized the problem of a lack of a scientific sound and internationally accepted approach to the estimation of measurement uncertainty. The need for an internationally accepted procedure for expression measurement uncertainty lead in 1981 that CIPM approving the brief outline recommendations submitted by a working group of representative from the major national standards laboratories. The International Organization of Standardization (ISO) was then given the task of developing a detailed guide applicable to all levels of accuracy from fundamental research to shop floor operations.

The responsibility for the preparation of such comprehensive document for this broad spectrum of measurements was assigned to a working group of the ISO technical Advisory Group on Metrology (ISO/TAG/WG3). This leads to the generation of "Guide to the Expression of Uncertainty in Measurement 1993. Since the publication of this guide, a number of simplified version standards were released such as "NIS 3003 (NAMAS. UK) and NIST (USA) which have the calculation, terminology and concepts that are consistent with the recommendation given in the Guide.
1.3 The Application of Measurement Uncertainties

There are number of applications of measurement uncertainty values. Its usefulness is summarized here.

1. For comparison with the previous measurement uncertainties to establish history, trend, or drift pattern. Thus, the stability of the equipment in terms of repeatability and reproducibility can be established.
2. Use it to determine the working tolerance of the equipment incompliant with 3X or 10X rule.
3. Selection of equipment using customer's specification with 3X/10X rule to determine the required uncertainty of instrument and compares with the existing uncertainties.
4. For establishing process (assembly and test) correlation between facility or equipment.

1.4 Definition and Symbol

The definition and symbol used in this document are defined here.

1.4.1 Definition

- **Value** - magnitude of a particular quantity generally expressed as a unit of measurement multiplied by a number such as length of a ruler is 150 cm.
- **True Value** - value consistent with definition of a given particular quantity. Note that the true value cannot be determined by measurement as all measurements have uncertainties.
- **Measurement** - set of operations having the objective of determining a value of a measurand.
- **Measurand** - particular quantity subject to measurement such as the relative humidity of a room.
- **Influence Quantity** - particular that is not the measurand but affects the result of measurement such as temperature of standard ruler used to measure the length.
- **Result of a measurement** - value attributed to a measurand obtained by measurement.
- **Accuracy of Measurement** - closeness of agreement between the result of measurement and true/reference value of the measurand. Note the term precision should not be used for "accuracy".
- **Repeatability** - closeness of the agreement between the results of successive measurements of the same measurand carried out under the same condition of measurement such as same operator, same DVM, and etc.
- **Reprocibility** - closeness of the agreement between the results of measurements of the same measurand carried out under **changed condition** of measurement such different method, different day, and etc.
- **Uncertainty of Measurement** - result of the evaluation aimed at characterizing the range within which the true value of a measurand is lied within given limits of confidence level.
- **Error** - result of a measurement minus a true value of the measurand.
• **Confidence Level** - the level of confidence associated with an interval within which a value is expected to lie. It is expressed as a percentage and is given by 100 multiplied by the probability that the value will fall within the specified interval. For uncertainty estimates, a confidence level of 95% is commonly used.

1.4.2 Symbol

- $a_i$ estimated sem-range of uncorrelated systematic component of uncertainty, probability distribution unknown where $i = 1, \ldots, N$.
- $a_d$ a systematic component of uncertainty that so dominates other contribution to uncertainty in magnitude that special consideration has to be given to its presence in calculating total uncertainty.
- $c_i$ sensitivity coefficient used to multiply input quantities $x_i$ to express them in terms of the output quantity $y$.
- $f$ Functional relationship between estimates of the measurand $y$ and the input estimates $x_i$ on which $y$ depends.
- $\partial f / \partial x_i$ Partial derivative with respect to input quantity $x_i$ of the functional relationship $f$ between the measureand and the input quantities.
- $s$ Experimental standard deviation that estimates the true standard deviation $\sigma$.
- $s(\bar{x})$ Experimental standard deviation of arithmetic mean $\bar{x}$.
- $t_p(v_{eff})$ Student $t$-factor of $v_{eff}$ degrees of freedom corresponding to a given probability $p$.
- $k$ Coverage factor used to calculate an expanded uncertainty $U$.
- $n$ number of repeated readings or observations.
- $N$ number of input estimates $x_i$ on which the measurand depends.
- $u(x_i)$ Standard uncertainty of input estimate $x_i$.
- $u(y_i)$ Combined standard uncertainty of output estimate $y_i$.
- $U$ Expand uncertainty of output estimate $y$ that provides a confidence interval $Y = y \pm U$.
- $v_{eff}$ Effective degrees of freedom of $u(y)$ used to obtain $t_p(v_{eff})$.
- $x_i$ Estimate of input quantity $X_i$.
- $y$ Estimate of the measurand $Y$. 
2.1 A bit of Statistics Theory

For any statistical data to be valid, the pre-requisition is to ensure the distribution of the population is normally distributed. For a given set of continuous observation, whether it is the measurement of wire length for a given production lot in the factory or the weight of boys in a classroom, a frequency plot for the measurement result will normally yield a symmetrical bell-shaped curve extending infinitely far at both positive and negative directions, which can be represented by a normal distribution. The normal distribution has the form as follow.

\[
F(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]
\]

where \(\mu\) is the mean or the average value for a given set of data and \(\sigma\) is the standard deviation which is the sum of the square of the deviation from the mean. \(F(x)\) is the frequency of occurrence or the probably density function of a given data having value \(x\). The integration of equation (2.1) from value \(x = -\infty\) to \(x = \infty\) will be equal to one which is also equal to the area under the bell-shaped curve. Thus,

\[
\int_{-\infty}^{\infty} F(x)dx = 1
\]

The mean (\(\mu\)) is defined as:

\[
\mu = \frac{\sum_{i=1}^{n} x_i}{n}
\]

where \(n\) is the number sample data and \(x_i\) is the sample value for \(i^{th}\) observation.

The variance (\(\sigma^2\)) is defined as:

\[
\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n - 1}
\]

Taking the square root of variance, it yields the standard deviation \(s\), which yields the equation (2.5).
\[ s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n-1}} \]  

(2.5)

For measurement process that involved taking more than one measurement range, the standard deviation of the value of measurand will not following equation (2.5). The pooled standard \( S_p \) shall be calculated, which shall follow equation (2.6).

\[ S_p = \sqrt{\frac{(n_1-1)(\sigma_{n_1-1})^2 + (n_2-1)(\sigma_{n_2-1})^2 + \cdots + (n_m-1)(\sigma_{n_m-1})^2}{(n_1+n_2+\cdots+n_m)-m}} \]  

(2.6)

where \( n_1, n_2, \ldots, n_m \) are the number of observation for particular measurement range and \( m \) is the number of measured range.

Many practical problems have the statistical answers based on the assumption that the distribution of the population is normal. If the sample data does not fit the normal distribution, modification is necessary. Otherwise, the statistical answer will not be true. As an example, the failure rate of a semiconductor device is exponentially distributed with respect to its lifetime, which is not a bell-shaped distribution. Therefore, modification is necessary for statistical assumption to be valid. In this case, it is usually modified to natural logarithm distribution before the sample data are statistically valid.

### 2.2 Source of Errors

In taking any measurement, there is error associated with it such as due to measurement instrument or the value measurand. If the error can be controlled within a certain limits of a certain confidence level, it is considered acceptable.

There are many sources of error and essentially they are from reference standard, material, human being, instrument, environment, and etc.

- Error in reference standard
- Error in reference materials
- Inaccuracy in working equipment
- Influence due to environmental condition such as thermal expansion coefficient of constant and temperature coefficient of semiconductor material.
- Personal bias in reading analogue instruments
- Finite instrument resolution or discrimination threshold
- Variation in repeated observations of the measureand under apparently identical conditions.

The error attributable to calibration or measurement instrument should be as small as possible. In most area of measurement, it should be no more than one third and preferably one tenth of the permissible error of the confirmed equipment.
2.3 Measurement Equation

The case of interest is where the quantity \( Y \) being measured, called the \textit{measurand}, is not measured directly, but is determined from \( N \) other quantities \( X_1, X_2, \ldots, X_N \) through a functional relation \( f \), often called the \textit{measurement equation}:

\[
Y = f(X_1, X_2, \ldots, X_N). \tag{2.7}
\]

Included among the quantities \( X_i \) are corrections (or correction factors), as well as quantities that take into account other sources of variability, such as different observers, instruments, samples, laboratories, and times at which observations are made (e.g., different days). Thus, the function \( f \) of equation (2.8) should express not simply a physical law but a measurement process, and in particular, it should contain all quantities that can contribute a significant uncertainty to the measurement result.

An estimate of the measurand or \textit{output quantity} \( Y \), denoted by \( y \), is obtained from equation (2.7) using \textit{input estimates} \( x_1, x_2, \ldots, x_N \) for the values of the \( N \) \textit{input quantities} \( X_1, X_2, \ldots, X_N \). Thus, the \textit{output estimate} \( y \), which is the result of the measurement, is given by

\[
y = f(x_1, x_2, \ldots, x_N). \tag{2.8}
\]

For example, as pointed out in the ISO \textit{Guide}, if a potential difference \( V \) is applied to the terminals of a temperature-dependent resistor that has a resistance \( R_0 \) at the defined temperature \( t_0 \) and a linear temperature coefficient of resistance \( b \), the power \( P \) (the measurand) dissipated by the resistor at the temperature \( t \) depends on \( V, R_0, b \), and \( t \) according to

\[
P = f(V, R_0, b, t) = \frac{V^2}{R_0[1 + b(t - t_0)]}. \tag{2.9}
\]

2.4 Classification of Uncertainty Components

The uncertainty of the measurement result \( y \) arises from the uncertainties \( u(x_i) \) (or \( u_i \) for brevity) of the input estimates \( x_i \) that enter equation (2.8). Thus, in the example of equation (2.9), the uncertainty of the estimated value of the power \( P \) arises from the uncertainties of the estimated values of the potential difference \( V \), resistance \( R_0 \), temperature coefficient of resistance \( b \), and temperature \( t \). In general, components of uncertainty may be categorized according to the method used to evaluate them.

2.4.1 Type A Evaluation

Type A error is the random error. This type of error is major contributed by series of measurement. The method of evaluation of uncertainty is by the statistical analysis of series of observations.

2.4.2 Type B Evaluation

Type B error is the systematic error. Examples of Type B error are readout resolution, hysteresis, finite precision arithmetic and rounding of reported values, effects of temperature and other environmental effect, uncertainty of the reference instrument,
and etc. The method of evaluation of uncertainty is by means other than the statistical analysis of series of observations.

2.5 Representation of Uncertainty Components

Each component of uncertainty, however evaluated, is represented by an estimated standard deviation, termed standard uncertainty with suggested symbol $u_i$, and equal to the positive square root of the estimated variance $u_i^2$. In generally there is type of standard uncertainty due to measurement, which are type A and type B respectively.

2.5.1 Standard Uncertainty of Type A

An uncertainty component obtained by a Type A evaluation is represented by a statistically estimated standard deviation $s_i$, equal to the positive square root of the statistically estimated variance $s_i^2$, and the associated number of degrees of freedom $v_i$. For such a component the standard uncertainty is $u_i = s_i$.

2.5.2 Standard Uncertainty of Type B

In a similar manner, an uncertainty component obtained by a Type B evaluation is represented by a quantity $u_j$. It may be considered an approximation to the corresponding standard deviation and is equal to the positive square root of $u_j^2$, obtained from an assumed probability distribution based on all the available information. Since the quantity $u_j^2$ is treated like a variance and $u_j$ like a standard deviation, for such a component the standard uncertainty is simply $u_j$. 
Chapter 3

Evaluation Uncertainty Components

3.1 Evaluating Type A Uncertainty Components

A Type A evaluation of standard uncertainty may be based on any valid statistical method for treating data. Examples are calculating the standard deviation of the mean of a series of independent observations; using the method of least squares to fit a curve to data in order to estimate the parameters of the curve and their standard deviations; and carrying out an analysis of variance ANOVA in order to identify and quantify random effects in certain kinds of measurements.

3.1.1 Mean and Standard Deviation

As an example of a Type A evaluation, consider an input quantity \( X_i \) whose value is estimated from \( n \) independent observations \( X_{i,k} \) of \( X_i \) obtained under the same conditions of measurement. In this case the input estimate \( x_i \) is usually the sample mean

\[
x_i = \bar{X}_i = \frac{1}{n} \sum_{k=1}^{n} X_{i,k}
\]

(3.1)

and the standard uncertainty \( u(x_i) \) to be associated with \( x_i \) is the estimated standard deviation of the mean

\[
u(x_i) = \frac{\sigma}{\sqrt{n}}
\]

\[
v = \sqrt{\frac{1}{n(n-1)} \sum_{k=1}^{n} (X_{i,k} - \bar{X}_i)^2}
\]

(3.2)

where \((n-1)\) is called degree of freedom \( v_i \).

3.2 Evaluating Type B Uncertainty Components

A Type B evaluation of standard uncertainty is usually based on scientific judgment using all of the relevant information available, which may include:

- previous measurement data,
- experience with, or general knowledge of, the behavior and property of relevant materials and instruments,
- manufacturer's specifications,
- data provided in calibration and other reports, and
- uncertainties assigned to reference data taken from handbooks.

Below are some examples of Type B evaluations in different situations, depending on the available information and the assumptions of the experimenter. Broadly speaking,
the uncertainty is either obtained from an outside source, or obtained from an assumed
distribution.

3.3 Uncertainty Obtained from an Outside Source

Normally the equipment calibrated from the outside source contains the value of
measurement uncertainty and the confidence interval level where this value is obtained.
Thus, the standard uncertainty of the equipment can be obtained from the procedures
specified in this sub-section below.

3.3.1 Multiple of a Standard Deviation

The procedure to obtain the standard deviation is as follows:
Convert an uncertainty quoted in a handbook, manufacturer's specification, calibration
certificate, etc., that is a stated multiple of an estimated standard deviation to a standard
uncertainty by dividing the quoted uncertainty by the multiplier.

3.3.2 Confidence Interval Level

The procedure to obtain the confidence interval is as follows:
Convert an uncertainty quoted in a handbook, manufacturer's specification, calibration
certificate, etc., that defines a "confidence interval" having a stated level of confidence,
such as 95 % or 99 %, to a standard uncertainty by treating the quoted uncertainty as if
a normal probability distribution had been used to calculate it (unless otherwise
indicated) and dividing it by the appropriate factor for such a distribution. These factors
are 1.960 and 2.576 for the two given confidence levels.

3.4 Uncertainty Obtained from an Assumed Distribution

In most circumstances, the measurement data should be normally distributed which has
mean $\bar{X}$ and standard deviation, $s$. This shall also mean that the data will fit the bell-
shaped probability density function. If the frequency distribution of the data is not
known, it is best to test them. There are two common methods used to test the
normality of the data. They are 1) by fitting the cumulative frequency data using
arithmetic normal distribution paper to see if a straight line can be drawn passing most
of the points and 2) by plotting the frequency chart to see if there is a bell-shaped
graph.

3.4.1 Normal Distribution of One Standard Deviation (1$\sigma$)

Model the input quantity in question by a normal probability distribution and estimate
lower and upper limits $a_-$ and $a_+$ such that the best estimated value of the input quantity
is $(a_+ + a_-)/2$ (i.e., the center of the limits) and there is 1 chance out of 2 (i.e., a 50 %
probability) that the value of the quantity lies in the interval $a_-$ to $a_+$. Then $u_j$ is
approximately $1.48 \, a$, where $a = (a_+ - a_-)/2$ is the half-width of the interval.

3.4.2 Normal Distribution of Two Standard Deviation (2$\sigma$)

Model the input quantity in question by a normal probability distribution and estimate
lower and upper limits $a_-$ and $a_+$ such that the best estimated value of the input quantity
is \((a_+ + a_-)/2\) (i.e., the center of the limits) and there are 2 chances out of 3 (i.e., a 67 % probability) that the value of the quantity lies in the interval \(a_+\) to \(a_-\). Then \(u_j\) is approximately \(a\), where \(a = (a_+ - a_-)/2\) is the half-width of the interval.

### 3.4.3 Normal Distribution of Three Standard Deviation (3σ)

If the quantity in question is modeled by a normal probability distribution, there are no finite limits that will contain 100 % of its possible values. However, plus and minus 3 standard deviations about the mean of a normal distribution corresponds to 99.73 % limits. Thus, if the limits \(a_+\) and \(a_-\) of a normally distributed quantity with mean \((a_+ + a_-)/2\) are considered to contain "almost all" of the possible values of the quantity, that is, approximately 99.73 % of them, then \(u_j\) is approximately \(a/3\), where \(a = (a_+ - a_-)/2\) is the half-width of the interval.

### 3.4.4 Uniform (Rectangular) Distribution

Estimate lower and upper limits \(a_-\) and \(a_+\) for the value of the input quantity in question such that the probability that the value lies in the interval \(a_-\) and \(a_+\) is, for all practical purposes, 100 %. Provided that there is no contradictory information, treat the quantity as if it is equally probable for its value to lie anywhere within the interval \(a_-\) to \(a_+\); that is, model it by a uniform (i.e., rectangular) probability distribution. The best estimate of the value of the quantity is then \((a_+ + a_-)/2\) with \(u_j = a\) divided by the square root of 3, where \(a = (a_+ - a_-)/2\) is the half-width of the interval.

### 3.4.5 Triangular Distribution

The rectangular distribution is a reasonable default model in the absence of any other information. But if it is known that values of the quantity in question near the center of the limits are more likely than values close to the limits, a normal distribution or, for simplicity, a triangular distribution, may be a better model.

Estimate lower and upper limits \(a_-\) and \(a_+\) for the value of the input quantity in question such that the probability that the value lies in the interval \(a_-\) to \(a_+\) is, for all practical purposes, 100 %. Provided that there is no contradictory information, model the quantity by a triangular probability distribution. The best estimate of the value of the quantity is then \((a_+ + a_-)/2\) with \(u_j = a\) divided by the square root of 6, where \(a = (a_+ - a_-)/2\) is the half-width of the interval.

### 3.5 Schematic Illustration of Probability Distributions

The following figure schematically illustrates the three distributions described above: normal, rectangular, and triangular. In the figures, \(\mu\) is the expectation or mean of the distribution, and the shaded areas represent \(\pm\) one standard uncertainty \(u\) about the mean. For a normal distribution, \(\pm u\) encompasses about 68 % of the distribution; for a uniform distribution, \(\pm u\) encompasses about 58 % of the distribution; and for a triangular distribution, \(\pm u\) encompasses about 65 % of the distribution.
Figure 3.1: The figures illustrate the probability distribution of normal, rectangular, and triangular functions.
Chapter 4

Combining Uncertainty Components

4.1 Calculation of Combined Standard Uncertainty

The combined standard uncertainty of the measurement result \( y \), designated by \( u_c(y) \) and taken to represent the estimated standard deviation of the result, is the positive square root of the estimated variance \( u_c^2(y) \) obtained from

\[
    u_c^2(y) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)
\]

Equation (4.1) is based on a first-order Taylor series approximation of the measurement equation \( Y = f(X_1, X_2, \ldots, X_N) \) given in equation (2.1) and is conveniently referred to as the law of propagation of uncertainty. The partial derivatives of \( f \) with respect to the \( X_i \) which is referred to as sensitivity coefficients, \( c_i \) are equal to the partial derivatives of \( f \) with respect to the \( X_i \) evaluated at \( X_i = x_i \); \( u(x_i) \) is the standard uncertainty associated with the input estimate \( x_i \); and \( u(x_i, x_j) \) is the estimated covariance associated with \( x_i \) and \( x_j \).

Take for example, the measurement equation is \( I = V/R \) where \( I \) is dependent on two variables \( V \) and \( R \). There are two sensitivity coefficients where one is \( C_V = \partial I / \partial V = 1/R \) and \( C_R = \partial I / \partial R = -V/R^2 \).

4.1.1 Simplified Forms

Equation (4.1) often reduces to a simple form in cases of practical interest. For example, if the input estimates \( x_i \) of the input quantities \( X_i \) can be assumed to be uncorrelated, then the second term vanishes. Further, if the input estimates are uncorrelated and the measurement equation is one of the following two forms, then equation (4.1) becomes simpler still.

**Measurement Equation:**

A sum of quantities \( X_i \) multiplied by constant \( a_i \)

\( Y = a_1X_1 + a_2X_2 + \ldots + a_NX_N \)

**Measurement Result:**

\( y = a_1x_1 + a_2x_2 + \ldots + a_Nx_N \)

**Combined Standard Uncertainty:**

\( u_c^2(y) = a_1^2u^2(x_1) + a_2^2u^2(x_2) + \ldots + a_N^2u^2(x_N) \)
Measurement Equation:
A product of quantities $X_i$, raised to powers $a, b, \ldots, p$, multiplied by a constant $A$.

$$Y = A X_1^a X_2^b \ldots X_N^p$$

Measurement Result:

$$y = A x_1^a x_2^b \ldots x_N^p$$

Combined Standard Uncertainty:

$$u_{c,r}^2(y) = a^2 u_{r}^2(x_1) + b^2 u_{r}^2(x_2) + \ldots + p^2 u_{r}^2(x_N)$$

Here $u_r(x_i)$ is the relative standard uncertainty of $x_i$ and is defined by $u_r(x_i) = u(x_i)/|x_i|$, where $|x_i|$ is the absolute value of $x_i$ and $x_i$ is not equal to zero; and $u_{c,r}(y)$ is the relative combined standard uncertainty of $y$ and is defined by $u_{c,r}(y) = u_c(y)/|y|$, where $|y|$ is the absolute value of $y$ and $y$ is not equal to zero.

### 4.1.2 Meaning of Uncertainty

If the probability distribution characterized by the measurement result $y$ and its combined standard uncertainty $u_c(y)$ is approximately normal (Gaussian), and $u_c(y)$ is a reliable estimate of the standard deviation of $y$, then the interval $y - u_c(y)$ to $y + u_c(y)$ is expected to encompass approximately 68% of the distribution of values that could reasonably be attributed to the value of the quantity $Y$ of which $y$ is an estimate. This implies that it is believed with an approximate level of confidence of 68% that $Y$ is greater than or equal to $y - u_c(y)$, and is less than or equal to $y + u_c(y)$, which is commonly written as $Y = y \pm u_c(y)$.

### 4.2 Coverage Factor

In general, the value of the coverage factor $k$ is chosen on the basis of the desired level of confidence to be associated with the interval defined by $U = ku_c$. Typically, $k$ is in the range 2 to 3. When the normal distribution applies and $u_c$ is a reliable estimate of the standard deviation of $y$, $U = 2 u_c$ (i.e., $k = 2$) defines an interval having a level of confidence of approximately 95%, and $U = 3 u_c$ (i.e., $k = 3$) defines an interval having a level of confidence greater than 99%.

The above general guideline for calculating the coverage factor is applicable only if following criteria are met.

1. If the uncertainty assessment involves only one Type A evaluation.
2. The number of sample reading, $n$ is greater than 2.
3. Value of Type A standard uncertainty is less than half value of the combined standard uncertainty.

Otherwise, the coverage factor should be calculated.

Coverage factor, $k$ can be accurately calculated by obtaining the an estimate of the effective degree of freedom, $v_{eff}$ of the combined standard uncertainty $u_c(y)$ by Welch-Satterwaite equation.
\begin{equation}
\nu_{\text{eff}} = \frac{\left[u_{c}(x)\right]^4}{\nu_1 + \left[u_{2}(x)\right]^4 + \cdots + \left[u_{n}(x)\right]^4} \tag{4.2}
\end{equation}

and then using \( t \)-distribution table to determine the coverage factor of a fixed confidence level.

**4.3 Relative Expanded Uncertainty**

In analogy with relative standard uncertainty \( u_r \) and relative combined standard uncertainty \( u_{c,r} \) defined above in connection with simplified forms of equation (4.2), the relative expanded uncertainty of a measurement result \( y \) is \( U_r = U/|y| \), \( y \) not equal to zero.

**4.4 Calculation of Measurement Uncertainty**

The sub-section detail out a step by step procedure to calculate the measurement uncertainty for the measurand.

**4.4.1 Calculation of Type A Standard Uncertainty**

The type A evaluation is used to obtain a value for repeatability or randomness of a measurement process exhibited on one particular occasion. For some measurements, the random component of uncertainty may not be significant in relation to other contributions to uncertainty. Nevertheless, the relevance of this type of evaluation should be determined.

The estimated standard deviation

\begin{equation}
S(\overline{X}) = \frac{\sigma_{n-1}}{\sqrt{n}} \tag{4.3}
\end{equation}

where \( \sigma_{n-1} \) is the standard deviation of the measured results and \( n \) is the number of repeated measurement.

Thus, the standard uncertainty of an input quantity, \( X_i \) evaluated by means of repeated measurement is obtained from

\begin{equation}
u(X_i) = S(\overline{X}) \tag{4.4}
\end{equation}

Pooled standard deviation is to be calculated using equation (4.5) if the Type A evaluation involves measurement over the range of the equipment being calibrated.

\begin{equation}
S_p = \sqrt{\frac{(n_1 - 1)(\sigma_{n_1 - 1})^2 + (n_2 - 1)(\sigma_{n_2 - 1})^2 + \cdots + (n_m - 1)(\sigma_{n_m - 1})^2}{(n_1 + n_2 + \cdots + n_m) - m}} \tag{4.5}
\end{equation}
where \((n_1 - 1), (n_2 - 1), \ldots (n_m - 1)\) are the degrees of freedom of the measurement range and \(m\) is the number of measurement range.

The estimated standard uncertainty shall then

\[ S(\bar{X}) = \frac{S_p}{\sqrt{n}} \]  

(4.6)

### 4.4.2 Calculation of Type B Standard Uncertainty

As it had mentioned, this type of uncertainty is referred as systematic uncertainty. They are usually comes from:

1. An uncertainty obtained from a calibration certificate where the level of confidence or coverage factor \((k)\) has been reported. Thus, the uncertainty can be treated as having normal distribution and its standard uncertainty \(u(x_i)\) is given by

\[ u(x_i) = \frac{a}{2} \text{ for 95% confidence level} \]  

(4.7)

where \(a\) is the measurement uncertainty of the equipment listed on the calibration certificate.

\[ u(x_i) = \frac{a}{3} \text{ for 99% confidence level} \]  

(4.8)

or

\[ u(x_i) = \frac{a}{k} \text{ is coverage factor is given} \]  

(4.9)

2. For the source of uncertainty, which does not have normal distribution behavior, in most of this case it would be assumed to be rectangular probability distribution, where its standard uncertainty \(u(x_i)\) is defined as

\[ u(x_i) = \frac{a}{\sqrt{3}} \]  

(4.10)

### 4.4.3 Calculation of Sensitivity Coefficient

For functional relationship of \(Y = f(X_1, X_2, \ldots, X_N)\) which has more than dependent variable, the sensitivity coefficient will not be equal to one. It depends on the partial differentiation of the \(Y\) with respect to \(X_1, X_2, \ldots, X_N\). Thus,

\[ C_1 = \frac{\partial f}{\partial X_1}, \quad C_2 = \frac{\partial f}{\partial X_2}, \quad C_3 = \frac{\partial f}{\partial X_3}, \quad \ldots, \quad C_N = \frac{\partial f}{\partial X_N}, \]  

(4.11)

As an illustration, let us calculate the measurement uncertainty of an area measured by a ruler. The area \((A) = \text{length (L) x width (W)}\).

Thus,
Sensitivity Coefficient for width \( C_w = \frac{\partial A}{\partial W} = L \)

and

Sensitivity Coefficient for length \( C_L = \frac{\partial A}{\partial L} = W \)

If the length \( L \) is 879 feet and width \( W \) is 650 feet, then the sensitivity coefficient for length is \( W \), which is 650 feet, whereas the sensitivity of width is \( L \), which is 879 feet.

### 4.4.4 Calculation of Combined Standard Uncertainty

Once the standard uncertainties, \( u(x_i) \), of the input quantities have been derived for both Type A and Type B evaluations, the standard uncertainty of the output quantity, \( y = f(x_1, x_2, \ldots, x_n) \), which is also called the combined standard uncertainty, \( u_c(y) \) is calculated using equation (4.12).

\[
\begin{align*}
\sum_{i=1}^{n} [c_i u_i(x)]^2 &= \sum_{i=1}^{n} (c_i u_i(x))^2 + \sum_{i=1}^{n} (c_i u_i(x))^2 + \ldots + \sum_{i=1}^{n} (c_i u_i(x))^2 \\
\end{align*}
\]

(4.12)

where \( c_1, c_2, \ldots, c_n \) are sensitivity coefficient of each error type. For independent related error type, the sensitivity coefficient is equal to 1.

### 4.5 Expanded Uncertainty

Although the combined standard uncertainty \( u_c \) is used to express the uncertainty of many measurement results, for some commercial, industrial, and regulatory applications (e.g., when health and safety are concerned), what is often required is a measure of uncertainty that defines an interval about the measurement result \( y \) within which the value of the measurand \( Y \) can be confidently asserted to lie. The measure of uncertainty intended to meet this requirement is termed expanded uncertainty, suggested symbol \( U \), and is obtained by multiplying \( u_c(y) \) by a coverage factor, suggested symbol \( k \). Thus \( U = ku_c(y) \) and it is confidently believed that \( Y \) is greater than or equal to \( y - U \), and is less than or equal to \( y + U \), which is commonly written as \( Y = y \pm U \).

### 4.6 Reporting the Results

After the value of expanded uncertainty has been calculated for a minimum confidence level of normally 95%. The value of the measurand and expanded uncertainty should be reported as \( y \pm U \) and accompanied by the following statement of confidence:

"The reported uncertainty is based on a standard uncertainty multiplied by a coverage factor \( k = 2 \) providing a level of confidence of 95%.

Report the results using "Uncertainty Budget Table" as shown in Appendix C."
4.7 Reliability of Uncertainty

If the reliability of the standard is given then the degree of freedom is depend on the relative uncertainty estimate, which is shown in table below. The table also shows that as the relative uncertainty estimate getting larger, the confidence level of the evaluation is getting lesser.

<table>
<thead>
<tr>
<th>Relative Uncertainty Estimate</th>
<th>$\nu$</th>
<th>Confidence Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>$\infty$</td>
<td>More confidence</td>
</tr>
<tr>
<td>10%</td>
<td>50</td>
<td>:</td>
</tr>
<tr>
<td>20%</td>
<td>12.5</td>
<td>:</td>
</tr>
<tr>
<td>25%</td>
<td>8</td>
<td>:</td>
</tr>
<tr>
<td>33%</td>
<td>4.5</td>
<td>:</td>
</tr>
<tr>
<td>50%</td>
<td>2</td>
<td>Less confidence</td>
</tr>
</tbody>
</table>

Table 4.1: Reliability of Uncertainty
Chapter 5

Summary of Procedure for Determining
The Measurement Uncertainty

The following steps serve as the guide to determine the measurement uncertainty of the measurand. Indeed they summarize the procedures mentioned in Chapter 3 and Chapter 4.

1. If possible the mathematical relationship between the input quantities and the output quantity. i.e. \( y = f(x_1, x_2, \ldots, x_N) \)
2. Identify all corrections that have to be applied to the measurement results of a measurand for the condition of measurement.
3. List systematic components of uncertainty associated with corrections and uncorrected systematic errors treated as uncertainties.
4. Seek prior experimental work or theory as a basis for assigning uncertainties and probability distributions to the systematic components of uncertainty.
5. Calculate the standard uncertainty for each component of uncertainty, obtained from Type B evaluation, using equation
   \[
   u(x_i) = \frac{a_i}{\sqrt{3}} \quad \text{or} \quad u(x_i) = \frac{(\text{expanded uncertainty})}{k}.
   \]
6. Use prior knowledge or make trial measurement and calculation to determine if there is going to be a random component of uncertainty that is significant compared with the effect of the listed systematic components of uncertainty.
7. If a random component of uncertainty is significant make repeated measurements to obtain the mean from equation
   \[
   \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.
   \]
8. Either calculate the standard deviation of the mean value from equation
   \[
   s(x_i) = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} \quad \text{or} \quad s(\bar{x}) = \frac{s(x_i)}{\sqrt{n}}, \quad \text{or refer to the results of previous repeatability measurements for a good estimate of } s(x_i) \text{ based on a larger number of readings}.
   \]
9. Even when the random component of uncertainty is not significant, where possible always check the instrument indication at least once to minimize operator recording error.
10. Derive the standard uncertainty for Type A evaluation using equation
    \[
    u(x_i) = s(\bar{x}).
    \]
11. Calculate the combined standard uncertainty for uncorrelated input quantities using equation
    \[
    u_c(y) = \sqrt{\sum_{i=1}^{N} c_i^2 u_i^2(x_i)} \equiv \sqrt{\sum_{i=1}^{N} u_i^2(y)} \quad \text{where } c_i \text{ is the partial derivative } \frac{\partial f}{\partial x} \text{ or known coefficient. Alternative use equation}
    \]
\[
\frac{u_c(y)}{|y|} = \sqrt{\sum_{i=1}^{N} \left| \frac{p_i u(x_i)}{|x_i|} \right|^2}
\]
where \( p \) are known positive or negative components in the functional relationship.

12. If correlation is suspected use the then the most straightforward approach is to add the standard uncertainties for the quantities before the above two equations shown in step 11.

13. Either calculate an expanded uncertainty from equation \( U = k u(y) \) by first calculating the effective degrees of freedom \( v_{\text{eff}} \) using equation (4.2) and \( t \)-distribution table shown in Appendix B to get the coverage factor.

Or

If there is a significant random contribution then use a coverage factor of 2 for estimated 95\% confidence level. The can be tested used three criteria listed in section 4.2.

14. Report the expanded uncertainty in the value of the measurand using format shown in Appendix C.

15. Advise the end user the significant of this measurement.
Chapter 6

Examples of Uncertainty Measurements

The following are examples of uncertainty statements as would be used in publication or correspondence. In each case, the quantity whose value is being reported is assumed to be a nominal 100g standard of mass $m_s$.

**Example 1**

$m_s = 100.02147$ g with a combined standard uncertainty (i.e., estimated standard deviation) of $u_c = 0.35$ mg. Since it can be assumed that the possible estimated values of the standard are approximately normally distributed with approximate standard deviation $u_c$, the unknown value of the standard is believed to lie in the interval $m_s \pm u_c$ with a level of confidence of approximately 68%.

**Example 2**

$m_s = (100.02147 \pm 0.00070)$ g, where the number following the symbol ± is the numerical value of an expanded uncertainty $U = ku_c$, with $U$ determined from a combined standard uncertainty (i.e., estimated standard deviation) $u_c = 0.35$ mg and a coverage factor $k = 2$. Since it can be assumed that the possible estimated values of the standard are approximately normally distributed with approximate standard deviation $u_c$, the unknown value of the standard is believed to lie in the interval defined by $U$ with a level of confidence of approximately 95%.

**6.1 One Dependent Variable**

**Example 3 Calibration of a Digital Multimeter**

A digital multimeter has been used to carry out a measurement of 100mV dc. A number of repeated measurements were made and the results are as follows:

<table>
<thead>
<tr>
<th>Repeated Measurement</th>
<th>Reading (mV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100.002</td>
</tr>
<tr>
<td>2</td>
<td>100.000</td>
</tr>
<tr>
<td>3</td>
<td>100.002</td>
</tr>
<tr>
<td>4</td>
<td>100.001</td>
</tr>
<tr>
<td>5</td>
<td>100.001</td>
</tr>
<tr>
<td>6</td>
<td>100.002</td>
</tr>
<tr>
<td>7</td>
<td>100.000</td>
</tr>
<tr>
<td>8</td>
<td>100.002</td>
</tr>
<tr>
<td>9</td>
<td>100.000</td>
</tr>
<tr>
<td>10</td>
<td>100.003</td>
</tr>
</tbody>
</table>

The following data were obtained by examining calibration certificates, manufacturer's specification and from previous experiments.
<table>
<thead>
<tr>
<th>Uncertainty Contribution</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>M.U. of the master equipment</td>
<td>0.0001mV</td>
</tr>
<tr>
<td>Drift since last calibration</td>
<td>0.002mV</td>
</tr>
<tr>
<td>Resolution of equipment</td>
<td>0.001mV</td>
</tr>
<tr>
<td>Accuracy</td>
<td>0.0005 mV</td>
</tr>
</tbody>
</table>

Question: Determine the measurement uncertainty of the 100mV measurement.

1. Standard uncertainty due to Type A evaluation is:

   \[ \bar{X} = 100.001mV \] and \( \sigma_{n-1} = 0.00114 \)

   The degree of freedom \( n-1 = 10-1 = 9 \)

   \[ u(x_i) = \frac{\sigma_{n-1}}{\sqrt{n}} = \frac{0.00114}{\sqrt{10}} = 0.000360mV \]

2. Standard uncertainty due to Type B evaluation are:
   a. Standard uncertainty due to M.U. of the master equipment

   \[ u(x_i) = \frac{a}{2} \text{ for 95\% confidence level} = \frac{0.0001}{2} = 0.00005 \text{ mV} \]
   b. Standard uncertainty due to drift since last calibration:

   \[ u(x_i) = \frac{a}{\sqrt{3}} = \frac{0.002}{\sqrt{3}} = 0.00115mV \]
   c. Standard uncertainty due to resolution of equipment:

   \[ u(x_i) = \frac{a}{2\sqrt{3}} = \frac{0.001}{2\times1.732} = 0.00029mV \]
   d. Standard uncertainty due to accuracy of the meter

   \[ u(x_i) = \frac{a}{\sqrt{3}} = \frac{0.0005}{1.732} = 0.00029mV \]

3. The combined standard uncertainty \( u_c(y) \) shall be:

   \[ u_c(y) = \sqrt{(0.00036)^2 + (0.0005)^2 + (0.00115)^2 + (0.00029)^2 + (0.00029)^2} \]

   \[ = 0.0012mV \]

4. The coverage factor, \( k \) is calculated first by calculating the effective degree of freedom.
\[ V_{\text{eff}} = \frac{0.00036^4}{9} + \frac{0.00005^4}{\infty} + \frac{0.00115^4}{\infty} + \frac{0.00029^4}{\infty} + \frac{0.00029^4}{\infty} \]

\[ = \frac{2.0736 \times 10^{-12}}{1.86624 \times 10^{-15}} = 1111 \]

Thus, from the table shown in Appendix B, at 95% confidence level, coverage factor, \( k = 2 \).

Alternatively using 3 criteria listed in sub-section 4.2 to test if \( k \) can be set equal to 2.

5. The Expanded uncertainty \( U = 0.0012 \times 2 = 0.0024 \text{ mV} \)

6. The reported measurement uncertainty is \( = 100.001 \pm 0.0024 \text{ mV} \)

The reported uncertainty is based on a standard uncertainty multiplied by a coverage factor \( k = 2 \), provided a level of confidence of 95.0% is used.

**Example 5**

**Calibration of a 10 k\( \Omega \) resistor by voltage intercomparision**

A long scale digital voltmeter is used to measure the voltage developed across a standard resistor and an unknown resistor of the same nominal value as the standard, when the series connected resistors are fed from a constant current source. The value of the unknown resistor, \( R_x \) is given by:

\[ R_x = (R_S + R_D + R_T) \frac{V_x}{V_S} \]

where \( R_S \) = calibration value for the standard resistor,
\( R_D \) = Drift in \( R_S \) since the last calibration.
\( R_T \) = Relative change due to the temperature of the oil bath.
\( V_x \) = voltage across \( R_x \).
\( V_S \) = Voltage across \( R_S \).

The calibration certificate for the standard resistor reported an uncertainty of \( \pm 15 \text{ ppm} \) at a confidence level of not less than 95% (\( k = 2 \)).

A correction was made for the estimated drift in the value \( R_S \). The uncertainty in this correction, \( R_D \), was estimated to have limits \( \pm 2.0 \text{ ppm} \).

The relative difference in resistance due to temperature variation in the oil bath as estimated to have limit of \( \pm 0.5 \text{ ppm} \).

The same voltmeter is used to measure \( V_x \) and \( V_S \). Although the uncertainty contribution will be correlated, the effect is to reduce the uncertainty and it is only
necessary to consider the relative difference in the voltmeter readings due to instability and resolution, which was estimated to have limit of ± 0.2 ppm for each reading.

Type A evaluation: Five measurements were made to record the departure from unity in the ratio $V_X/V_S$ in ppm. The readings were as follows:

$$+10.4, +10.7, +10.6, +10.3, +10.5$$

The mean value $\bar{V} = +10.5$ ppm and $u(V) = s(\bar{V}) = 0.158/\sqrt{5} = 0.0706$ ppm.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Source of Uncertainty</th>
<th>Value ± ppm</th>
<th>Probability Distribution</th>
<th>Divisor</th>
<th>$C_i$</th>
<th>$u_i(R_x)$ ± ppm</th>
<th>$v_i$ or $v_{eff}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_S$</td>
<td>Calibration of standard resistor</td>
<td>1.5</td>
<td>normal</td>
<td>2.0</td>
<td>1.0</td>
<td>0.75</td>
<td>∞</td>
</tr>
<tr>
<td>$R_D$</td>
<td>Uncorrected drift since last calibration</td>
<td>2.0</td>
<td>rectangular</td>
<td>$\sqrt{3}$</td>
<td>1.0</td>
<td>1.155</td>
<td>∞</td>
</tr>
<tr>
<td>$R_r$</td>
<td>Effect of temperature of oil bath</td>
<td>0.5</td>
<td>rectangular</td>
<td>$\sqrt{3}$</td>
<td>1.0</td>
<td>0.289</td>
<td>∞</td>
</tr>
<tr>
<td>$V_S$</td>
<td>Voltage across $R_S$</td>
<td>0.2</td>
<td>rectangular</td>
<td>$\sqrt{3}$</td>
<td>1.0</td>
<td>0.115</td>
<td>∞</td>
</tr>
<tr>
<td>$V_X$</td>
<td>Voltage across $R_X$</td>
<td>0.2</td>
<td>rectangular</td>
<td>$\sqrt{3}$</td>
<td>1.0</td>
<td>0.115</td>
<td>∞</td>
</tr>
<tr>
<td>$V$</td>
<td>Repeatability</td>
<td>0.071</td>
<td>normal</td>
<td>1.0</td>
<td>1.0</td>
<td>0.071</td>
<td>4</td>
</tr>
<tr>
<td>$u_i(R_x)$</td>
<td>Combined uncertainty</td>
<td>normal</td>
<td>Normal ($k = 2$)</td>
<td>1.418</td>
<td>&gt; 500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U$</td>
<td>Expanded uncertainty</td>
<td></td>
<td>Normal ($k = 2$)</td>
<td>2.836</td>
<td>&gt; 500</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Report Result:**

Measured value of 10kΩ resistor is: 10,000.11Ω ± 0.03Ω

The reported uncertainty is based on a standard uncertainty multiplied by a coverage factor $k =2$, which provides a confidence level of approximately 95%.

### 6.2 More than One Dependent Variable

#### Example 4

Calculate the measurement uncertainty of the area of a swimming pool which is designed with length (L) 880 feet and width (W) 650 feet using a standard ruler that has standard uncertainty of 6.5 inches and the following data. Coverage factor of two is assumed for this calculation.

<table>
<thead>
<tr>
<th>Measurement Type</th>
<th>Number of Repeat Measurement</th>
<th>Mean Value</th>
<th>Standard Deviation</th>
<th>Standard Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (L)</td>
<td>5</td>
<td>879.3 feet</td>
<td>0.72 feet</td>
<td>0.321 feet</td>
</tr>
<tr>
<td>Width (W)</td>
<td>7</td>
<td>650.3 feet</td>
<td>0.98 feet</td>
<td>0.370 feet</td>
</tr>
</tbody>
</table>
The sensitivity coefficient of length \( C_L = \frac{\partial A}{\partial L} = \frac{\partial (LW)}{\partial L} = W = 650.3 \text{ feet} \) and the sensitivity coefficient of width \( C_w = \frac{\partial A}{\partial W} = \frac{\partial (LW)}{\partial W} = L = 879.3 \text{ feet} \)

The combined standard uncertainty is

\[
\left( C_w u(x_w) \right)^2 + \left( C_L u(x_L) \right)^2 + \left[ \frac{6.5}{(12\sqrt{3})} \right]^2 \right)^{1/2} \\
= \left[ 879.3 \times 0.370 \right]^2 + \left[ 650.3 \times 0.321 \right]^2 + \left[ \frac{6.5}{12\sqrt{3}} \right]^2 \right)^{1/2} \\
= 388.8 \text{ feet}^2
\]

Thus, the expanded uncertainty of the area of swimming pool is \( 2 \times 388.8 = 777.7 \text{ feet}^2 \).

The standard uncertainty of the area of swimming pool is \( 571,808.8 \pm 777.7 \text{ feet}^2 \) a level of confidence of 95.0% is used.
Chapter 7

Workshop

Workshop 1
Calculate the uncertainty of measurement for Analysis of Total Dissolved Solids.

The Total Dissolved Solids Meter has three ranges i.e. 50ppm, 100 ppm and 1,000ppm. According to the manufacturer's data sheet, the accuracy of the meter is ± 3% of full-scale deflection. The meter can give the smallest reading to 0.5ppm.

Calibration of the meter was carried out using standard solution 50 ppm at the 50ppm range. The standard solution has an accuracy of ± 1%.

10 readings were recorded:
49.4, 50.2, 50.2, 49.4, 50.0, 50.1 49.8, 49.8, 50.0, 50.2

Workshop 2
Using the data provided in workshop 1, calculate the number of effective measurement. Another word is the optimum repetition.

Workshop 3
Calibration of measurement uncertainty of your wristwatch with a stopwatch

Using the stopwatch provided to you, calculate the measurement uncertainty of your wristwatch by measuring 10, 20, 30, 45, 60 seconds intervals with 3 repeated reading for each interval.

The uncertainty of the stopwatch is ±0.005s at 95.0% confidence level. The resolution of stopwatch is ±0.01s and the resolution of your wristwatch is obtained from the watch that you used.
Homework
A current of 10A is measured using a current shunt and a voltmeter. The specifications of the instrument used are;

1. Current Shunt Specification
   - Current: 10 A
   - Resistance: 0.01Ω
   - Calibration report
     Resistance $R = 0.010088\ \Omega$ measured at 10A, 23°C.
     Relative expanded uncertainty: $\pm 8 \times 10^{-4}$ at coverage factor $k = 2$
     Temperature coefficient in the range 15°C to 30°C; 60ppm/K
     The relative expanded uncertainties can be converted into expanded uncertainty by multiplying by $R$.

2. Voltmeter Specification
   - All uncertainty specification applied for one year after calibration when operated in a temperature of 15°C to 40°C and relative humidity of up to 80%.
   - DC voltage function specification

<table>
<thead>
<tr>
<th>Range</th>
<th>Full Scale</th>
<th>Uncertainty $(% \text{ of reading } + \text{ number of counts})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200mV</td>
<td>199.99mV</td>
<td>0.030 +2</td>
</tr>
</tbody>
</table>

3. Measurement Record at room temperature (23 ± 5)°C, humidity: (55 ± 10)% RH.

<table>
<thead>
<tr>
<th>Reading No.</th>
<th>Voltage (mV)</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>2</td>
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<tr>
<td>4</td>
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<td>100.63</td>
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<tr>
<td>7</td>
<td>100.60</td>
</tr>
<tr>
<td>8</td>
<td>100.68</td>
</tr>
<tr>
<td>9</td>
<td>100.76</td>
</tr>
<tr>
<td>10</td>
<td>100.65</td>
</tr>
</tbody>
</table>

Question:

Calculate the expanded uncertainty for the current meter.
Answer for the Homework

The current is a function of voltage and resistance. Thus, from Ohm's Law \( I = \frac{V}{R} \), where \( V \) is the voltage and \( R \) is the resistance.

The main source of uncertainty in the measurement is:

1. Uncertainty due to the measurement of voltage which consists of the random variability of the measured voltage value and the measurement uncertainty of the voltmeter. The effect of the room temperature on voltmeter has already been taken into account in the error limits.
2. Uncertainty due to current shunt resistance which consists of the calibrated resistance value and the resistance change due to temperature effect.

Calculation of Type A Standard Uncertainty

The standard uncertainty \( u_1(V) \) of Type A random error due to measured voltage of mean 100.72mV and standard deviation 10.75x10\(^{-2}\)mV is \( u_1(V) = \frac{10.75 \times 10^{-2}}{\sqrt{10}} = 3.40 \times 10^{-5} \)V. 10 are the number of measurement.

Calculation of Type B Standard Uncertainty

The standard uncertainty of voltmeter from specification is \( u_2(V) = (0.030\% \times 100.72mV + 0.02mV) / \sqrt{3} = 2.90 \times 10^{-5} \)V with degree of freedom \( v_2 \) equals to \( \infty \).

The standard uncertainty of current shunt resistance as reported is \( u_3(R) = 8.00 \times 10^{-4} \times 0.010088 / 2 = 4.04 \times 10^{-6} \)Ω with degree of freedom \( v_3 \) equals to \( \infty \).

From calibration report, the standard uncertainty of the shunt resistance due to temperature effect of room temperature \( \Delta t = 5^\circ C \) is \( u_4(R) = \frac{6 \times 10^{-6} \times 5 \times 0.010088}{\sqrt{3}} = 1.75 \times 10^{-6} \)Ω with degree of freedom \( v_4 \) equal to \( \infty \).

The current \( I \) is equal to \( \frac{V}{R} = \frac{100.72 \times 10^{-3}}{0.010088} = 9.984A \).

Thus the combined standard uncertainty for current measurement is

\[
u_c(I) = \sqrt{C_v^2[u_1^2(V) + u_2^2(V)] + C_R^2[u_3^2(R) + u_4^2(R)]}
\]

where

\[C_v = \frac{\partial I}{\partial V} = \frac{1}{R} = \frac{1}{0.010088} = 99.128 \Omega^{-1}\]

and
\[
C_R = \frac{\partial I}{\partial R} = -\frac{V}{R^2} = -\frac{100.72 \times 10^{-3}}{0.01008^2} = -989.7V\Omega^{-2}
\]

then

\[
u_c(I) = \sqrt{99.128^2[(3.4 \times 10^{-5})^2 + (2.90 \times 10^{-5})^2] + (-989.7)^2[(4.04 \times 10^{-6})^2 + (1.75 \times 10^{-6})^2]}
= \sqrt{3.86 \times 10^{-5}} = 6.2 \times 10^{-3} A
\]

The effective degree of freedom \( v_{\text{eff}} \).

\[
v_{\text{eff}} = \frac{[6.2 \times 10^{-3}]^4}{9} + \frac{[99.128 \times 3.4 \times 10^{-5}]^4}{\infty} + \frac{[-989.7 \times 4.04 \times 10^{-6}]^4}{\infty} + \frac{[-989.7 \times 1.75 \times 10^{-6}]^4}{\infty}
= 103.
\]

From the student \( t \)-distribution table, for degrees of freedom \( v_{\text{eff}} > 100 \), at 95% confidence level, the \( t \)-factor is 1.96.

Therefore, the coverage factor \( k = 1.96 \)

and

The expanded uncertainty \( U \) is 1.96x6.2x10^{-3} = 0.012A.
Appendix A

Background

A measurement result is complete only when accompanied by a quantitative statement of its uncertainty. The uncertainty is required in order to decide if the result is adequate for its intended purpose and to ascertain if it is consistent with other similar results.

International and U.S. Perspectives on Measurement Uncertainty

Over the years, many different approaches to evaluating and expressing the uncertainty of measurement results have been used. Because of this lack of international agreement on the expression of uncertainty in measurement, in 1977 the International Committee for Weights and Measures (CIPM, Comité International des Poids et Mesures), the world's highest authority in the field of measurement science (i.e., metrology), asked the International Bureau of Weights and Measures (BIPM, Bureau International des Poids et Mesures), to address the problem in collaboration with the various national metrology institutes and to propose a specific recommendation for its solution. This led to the development of Recommendation INC-1 (1980) by the Working Group on the Statement of Uncertainties convened by the BIPM, a recommendation that the CIPM approved in 1981 and reaffirmed in 1986 via its own Recommendations 1 (CI-1981) and 1 (CI-1986):

Recommendation INC-1 (1980)

Expression of Experimental Uncertainties

1. The uncertainty in the result of a measurement generally consists of several components, which may be grouped into two categories according to the way in which their numerical value is estimated.
   Type A: Those uncertainties that are evaluated by statistical methods
   Type B: Those uncertainties that are evaluated by other mean such given by another calibration laboratory.

   There is no simple correspondence between the classification for categories A or B and the previously it is classified into "random" and "systematic" uncertainties. The term "systematic uncertainty" can be misleading and should be avoided.

   Any detailed report of uncertainty should consist of a complete list of the components, specifying for each method used to obtain its numerical value.

2. The components in category A are characterized by the estimated variances \( s_i^2 \) (or the estimated "standard deviations" \( s_i \)) and the number of degrees of freedom \( v_i \), where it is appropriate, the covariance should be given.

3. The components in category B should be characterized by quantities, \( u_j^2 \), which may be considered approximation to the corresponding variance, the existence of it is assumed. The quantities, \( u_j^2 \) may be treated like variances and the quantities \( u_j \) like standard deviations. Where appropriate, the covariance should be treated in a similar way.
4. The combined uncertainty should be characterized by the numerical value obtained by applying the usual method for the combination of variances. The combined uncertainty and its components should be expressed in the form of "standard deviations."

5. If for particular application, it is necessary to multiply the combined uncertainty by an overall uncertainty, then the multiplying factor must always be stated.

The above recommendation, INC-1 (1980), is a brief outline rather than a detailed prescription. Consequently, the CIPM asked the International Organization for Standardization (ISO) to develop a detailed guide based on the recommendation because ISO could more easily reflect the requirements stemming from the broad interests of industry and commerce. The ISO Technical Advisory Group on Metrology (TAG 4) was given this responsibility. It in turn established Working group 3 and assigned it the following terms of reference:

To develop a guided document based upon the recommendation of the BIPM Working Group on the Statement of Uncertainties which provides rules on the expression of measurement uncertainty for use within standardization, calibration, laboratory accreditation, and metrology services;

The purpose of such document is:

- to promote full information on how uncertainty statements are arrived at;
- to provide a basis for the international comparison of measurement results.

The Guide to the Expression of Uncertainty in Measurement

The end result of the work of ISO/TAG 4/WG 3 is the 100-page Guide to the Expression of Uncertainty in Measurement (or GUM as it is now often called). It was published in 1993 (corrected and reprinted in 1995) by ISO in the name of the seven international organizations that supported its development in ISO/TAG 4:

| BIPM | Bureau International des Poids et Mesures |
| IEC  | International Electrotechnical Commission |
| IFCC | International Federation of Clinical Chemistry |
| ISO  | International Organization for Standardization |
| IUPAC| International Union of Pure and Applied Chemistry |
| IUPAP| International Union of Pure and Applied Physics |
| OIML | International Organization of Legal Metrology |

The focus of the ISO Guide or GUM is the establishment of "general rules for evaluating and expressing uncertainty in measurement that can be followed at various levels of accuracy and in many fields--from the shop floor to fundamental research." As a consequence, the principles of the GUM are intended to be applicable to a broad spectrum of measurements, including those required for:

- maintaining quality control and quality assurance in production;
- complying with and enforcing laws and regulations;
- conducting basic research, and applied research and development, in science and engineering;
• calibrating standards and instruments and performing tests throughout a national measurement system in order to achieve traceability to national standards;
• developing, maintaining, and comparing international and national physical reference standards, including reference materials.

Wide Acceptance of the GUM
The GUM has found wide acceptance in the United States and other countries. For example:

The GUM method of evaluating and expressing measurement uncertainty has been adopted widely by U.S. industry as well as companies abroad. The National Conference of Standards Laboratories (NCSL), which has some 1500 members, has prepared and widely distributed Recommended Practice RP-12, Determining and Reporting Measurement Uncertainties, based on the GUM. ISO published the French translation of the GUM in 1995, German and Chinese translations were also published in 1995, and an Italian translation was published in 1997. Translations of the GUM into Estonian, Hungarian, Italian, Japanese, Spanish, and Russian have been completed or are well underway.

GUM methods have been adopted by various regional metrology and related organizations including:

<table>
<thead>
<tr>
<th>NORAMET</th>
<th>North American Collaboration in Measurement Standards</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAVLAP</td>
<td>National Voluntary Laboratory Accreditation Program</td>
</tr>
<tr>
<td>A2LA</td>
<td>American Association for Laboratory Accreditation</td>
</tr>
<tr>
<td>EUROMET</td>
<td>European Collaboration in Measurement Standards</td>
</tr>
<tr>
<td>EUROLAB</td>
<td>A focus for analytic chemistry in Europe</td>
</tr>
<tr>
<td>EAL</td>
<td>European Cooperation for Accreditation of Laboratories</td>
</tr>
</tbody>
</table>

Moreover, the GUM has been adopted by NIST and most of NIST's sister national metrology institutes throughout the world, such as the National Research Council (NRC) in Canada, the National Physical Laboratory (NPL) in the United Kingdom, and the Physikalisch-Technische Bundesanstalt in Germany.


It is noteworthy that NIST's adoption of the GUM approach to expressing measurement uncertainty was done with considerable forethought. Although quantitative statements of uncertainty had accompanied most NIST measurement results, there was never a uniform approach at NIST to the expression of uncertainty. Recognizing that the use of a single approach within NIST instead of a variety of approaches would simplify the interpretation of NIST outputs, and that U.S. industry was calling for a uniform method of expressing measurement uncertainty, in 1992 then NIST Director J. W. Lyons appointed a NIST Ad Hoc Committee on Uncertainty Statements to study the issue. In particular, the Ad Hoc committee was asked to ascertain if the GUM approach would meet the needs of NIST's customers. The
conclusion was that it most definitely would, and a specific policy for the implementation of the GUM approach at NIST was subsequently adopted.

NIST Technical Note 1297 (TN 1297) (see the Bibliography for full citation) was prepared by two members of the Ad Hoc Committee, who also played major roles in the preparation of the GUM. (The policy, "Statement of Uncertainty Associated with Measurement Results," was incorporated in the NIST Administrative Manual and is included as Appendix C in TN 1297.) TN 1297 has in fact found broad acceptance. To date, over 30,000 copies have been distributed to NIST staff and in the United States at large and abroad -- to metrologists, scientists, engineers, statisticians, and others who are involved with measurement in some way.

**Joint Committee for Guides on Metrology (JCGM)**

Most recently, a new international organization has been formed to assume responsibility for the maintenance and revision of the GUM and its companion document the VIM (see the Bibliography for a brief discussion of the VIM). The name of the organization is JCGM and its members are the seven international organizations listed above: BIPM, IEC, IFCC, ISO, IUPAC, IUPAP, and OIML. ISO/TAG 4 has been reconstituted as the Joint ISO/IEC TAG, Metrology, and will focus on metrological issues internal to ISO and IEC as well as represent ISO and IEC on the JCGM.
## Appendix B

### $t$-distribution for Degree of Freedom $\nu$

<table>
<thead>
<tr>
<th>Degree of freedom, $\nu$</th>
<th>Fraction $p$ in percent</th>
<th>68.27(*)</th>
<th>90</th>
<th>95</th>
<th>95.45(*)</th>
<th>99</th>
<th>99.73(*)</th>
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<tr>
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<td>2.576</td>
<td>3.000</td>
<td></td>
</tr>
</tbody>
</table>

(*) For quantity $z$ described by normal distribution with expectation $\mu_z$ and standard deviation $\sigma$, the interval $\mu_z \pm k\sigma$ encompasses $p = 68.27$, 95.45 and 99.73 percent of the distribution for $k = 1, 2, \text{ and } 3$. 

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Appendix C

Uncertainty Budget Table

Calibration Description: _____________________________________ U.U.T Identification Number: _____________________________

Reference Equipment: ________________________________________ Date of Calibration/Measurement: _________________________

<table>
<thead>
<tr>
<th>Source of Uncertainty</th>
<th>Type</th>
<th>Uncertainty Value</th>
<th>Probability Distribution</th>
<th>Divisor</th>
<th>Sensitive Coefficient</th>
<th>Standard Uncertainty</th>
<th>Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repeatability</td>
<td>A</td>
<td></td>
<td>Normal</td>
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<td></td>
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</tr>
<tr>
<td>Reference Standard</td>
<td>B</td>
<td></td>
<td>Normal</td>
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<tr>
<td>Resolution or (ref)</td>
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<td></td>
<td>Rectangular</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>Rounding Off (U.U.T)</td>
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<tr>
<td>Known or built-in Uncertainty</td>
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<td>Rectangular</td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Combined Standard Uncertainty, \( u_c(y) = \sqrt{c_1u_1(x)^2} + c_2u_2(x)^2 + \ldots + c_nu_n(x)^2 = \)

If standard criterion NOT satisfied, calculate the Effective Degree of Freedom \( v_{eff} = \frac{[u_c(y)]^4}{v_1} + \frac{[u_c(y)]^4}{v_2} + \ldots + \frac{[u_c(y)]^4}{v_n} \)

Coverage factor, \( k \)

Expand Uncertainty, \( U_y \)
Bibliography

[1] Sources
The information on evaluating and expressing measurement uncertainty within this reference is adapted from NIST Technical Note 1297. The publication TN 1297, prepared by B. N. Taylor and C. E. Kuyatt and entitled Guidelines for Evaluating and Expressing the Uncertainty of NIST Measurement Results, is in turn based on the comprehensive International Organization for Standardization (ISO) publication, Guide to the Expression of Uncertainty in Measurement.

[2] Background
Background information on the development of the ISO Guide, its worldwide adoption, NIST TN 1297, and the NIST policy on expressing measurement uncertainty is given in the section International and U.S. perspectives on measurement uncertainty.

[3] Related Information
Note also that a companion document to the ISO Guide, or GUM, was prepared by ISO/TAG 4/WG 1 and published in 1993 by ISO in the name of the same seven international organizations in whose name the GUM was published. It is entitled International Vocabulary of Basic and General Terms in Metrology, or VIM, and gives the definitions of many important terms relevant to the field of measurement (a few of these definitions are reprinted in the GUM and in TN 1297).
