Chapter 15

Active Filter Circuits

15.0 Introduction

Filter is circuit that capable of passing signal from input to output that has frequency within a specified band and attenuating all others outside the band. This is the property of selectivity.

They are four basic types of filters. They are low-pass, high-pass, band-pass, and band-stop. The all-pass filter circuit that can be designed.

The basic filter is achieved by with various combinations of resistors, capacitors, and sometimes inductors. It is called passive filter. Active filters use transistors or operational amplifier and RC circuit to provide desired voltage gains or impedance characteristics. Inductance is not preferred for active filter design because it is least ideal, bulky, heavy, and expensive and does not lend itself to IC-type mass production.

Each type of filter response can be tailored by circuit component values that have Butterworth, Chebyshev, or Bessel characteristics. Each of these characteristics is identified by the shape of its response curve and each has an advantage in certain application.

Butterworth characteristic has very flat amplitude in the pass band and a roll-off rate of -20dB/decade/pole. The phase response is not linear. However, the phase shift of the signals passing through the filter varies nonlinearly with frequency. Therefore, a pulse applied to a filter with Butterworth response will cause overshoots on the output because each frequency component of the pulse's rising and falling edges experiences a different time delay.

Chebyshev has characteristic response that roll-off greater than -20dB/decade/pole. The circuit has characteristic of overshoot and ripple response in the pass band.

Bessel has a linear phase characteristic, which shall mean that the phase shift increases linearly with frequency. Thus, Bessel response is used for filtering pulse waveform without distorting the shape of the waveform.
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The damping factor $\zeta$ of the active filter determines which characteristic the filter exhibits. Refer to Fig. 15.6, the circuit has a RC filter element at positive feedback and negative feedback circuit, which comprises resistors $R_1$ and $R_2$.

The characteristics of Butterworth, Chebyshev and Bessel filters are shown in Fig. 15.1.

**Figure 15.1**: The characteristic of Butterworth, Chebyshev and Bessel filters

### 15.1 Transfer Function

Filters are implemented with devices exhibiting frequency-dependent characteristic like inductor and capacitor. The behavior of a circuit is uniquely characterized by its transfer function $T(s)$. Using simple laws such as KVL, KCL, and the superposition theorem, the transfer function $T(s)$, which is the ratio of output voltage or current and input voltage or current, is mathematically expressed for voltage as

$$T(s) = \frac{V_{out}(s)}{V_{in}(s)} \quad (15.1)$$

Once the function $T(s)$ is known, the output voltage response $V_{out}(t)$ to a given input voltage response $V_{in}(t)$ can be determined from the inverse Laplace’s function containing the transfer function and the input voltage, which is expressed as

$$V_{out}(t) = \mathcal{L}^{-1}\{T(s)V_{in}(s)\} \quad (15.2)$$
The transfer function $T(s)$ can be written as a polynomial as shown in equation (15.3).

$$T(s) = \frac{N(s)}{D(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \ldots + b_0s^0 + b_n}{s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0}$$  \hspace{1cm} (15.3)$$

$N(s)$ and $D(S)$ are suitable polynomials of $s$ real coefficient and degrees $m$ and $n$. The function is a rational function of $s$, which is strictly proper when $m \leq n-1$.

The degree $n$ in the denominator determined the order of the filter such as first order, second order, etc. The roots of equation $N(s) = 0$ and $D(s) = 0$ are called respectively the zeros and the poles of the transfer function $T(s)$. They denote as $z_1, z_2, \ldots, z_n$ and $p_1, p_2, \ldots, p_n$. Factoring out $N(s)$ and $D(s)$ in terms of their respective roots, the transfer function can be written as

$$T(s) = \frac{a_m}{b_n} \cdot \frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$  \hspace{1cm} (15.4)$$

where $a_m/b_n$ is the scaling factor. Roots are also referred as critical or characteristic frequencies. Roots can be real or complex. When zero or poles are complex, they occur in conjugate pairs. For instance, if $p_k = \sigma_k + j\omega_k$ is a pole then $p_k^* = \sigma_k - j\omega_k$ is also a pole. Roots are conveniently visualized as point in the complex plane or $s$ plane: $\sigma_k$ is plotted against horizontal or real axis, which is calibrated in nepers per second $N_p/s$: $\omega_k$ is plotted against vertical or imaginary axis, which calibrated in radian per second. In the plot, a zero is represented as “o” and a pole is represented as “x”.

**Example 15.1**
Find the pole-zero plot of the circuit shown in the Figure.
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**Solution**

The Transfer function is

\[ T(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{sRC}{s^2LC + sRC + 1} = \frac{R \cdot s}{L \cdot s^2 + (R/L)s + 1/LC}. \]

Substitute the known values for L, C, and R, the transfer function becomes \( T(s) = \frac{2 \times 10^3 s}{[s - (-1 + j2)10^3]\times[s - (-1 - j2)10^3]} \). Thus, the function has a scaling factor of \( 2 \times 10^3 \) V/V, a zero at the origin, and conjugate pole pair at \(-1 \pm j2\) complex KNp/s. Its pole-zero plot is shown below.

![Pole-Zero Plot](image)

**15.2 General Two-Pole Active Filter**

A general two-pole active filter is shown in Fig. 15.2 with \( Y_1 \) through \( Y_4 \) are admittances and with an ideal voltage follower.

![General Two-Pole Active Filter](image)

**Figure 15.2:** An unity gain general two-pole active filter

A KCL equation at node \( V_a \) shall be
(\(V_{\text{in}} - V_a\))Y_1 = (V_a - V_b)Y_2 + (V_a - V_{\text{out}})Y_3 \tag{15.5}

A KCL equation at node \(V_b\) produces

\((V_a - V_b)Y_2 = V_bY_4 \tag{15.6}\)

Since \(V_b\) is also equal to \(V_{\text{out}}\) therefore

\[V_a = V_b \left( \frac{Y_2 + Y_4}{Y_2} \right) = V_{\text{out}} \left( \frac{Y_2 + Y_4}{Y_2} \right) \tag{15.7}\]

Thus, substitute equation (15.7) into equation (15.5) and multiply it by term \(Y_2\), it yields the transfer function \(T(s)\) for the filter, which is

\[T(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{Y_1Y_2}{Y_1Y_2 + Y_4(Y_1 + Y_2 + Y_3)} \tag{15.8}\]

For non-unity gain filter, the transfer function of the general two-pole filter follows equation (15.9).

\[T(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{A_V Z_2Z_4}{Z_1Z_2 + Z_2Z_3 + Z_3Z_4 + Z_1Z_3 + Z_1Z_4 (1 - A_V)} \tag{15.9}\]

\(A_V\) is the pass-band gain of the filter, which is defined as \(A_V = 1 + \frac{R_1}{R_2}\) and \(Z_1\), \(Z_2\), \(Z_3\), and \(Z_4\) are impedances. The equation can be derived using KCL law and the circuit shown in Fig. 15.3.

**Figure 15.3:** A non-unity gain general two-pole active filter
15.2 Low-Pass Filter

A passive low-pass filter is one that can sufficiently attenuate all frequencies above a certain frequency named as critical frequency $f_c$ and passes all frequencies below $f_c$ value. It has a basic filter element consists of a RC circuit, which is shown in Fig. 15.4. Since it has a RC element, it is a single-pole filter. The expected output signals with respect to the input signals of various frequencies below, equal, and above critical frequency are shown in Fig. 15.5. For the case where input frequency $f$ is much greater than critical frequency $f_c$, the output at the filter is approximately zero volt.

![Figure 15.4: A basic low-pass passive filter](image)

![Figure 15.5: The expected output signal of low-pass filter with input of various frequency](image)
The transfer function $T(s)$ of the filter shall be

$$T(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{sC + R} = \frac{1}{1 + sRC}$$  \hspace{1cm} (15.10)

The magnitude of the function is $|T(s)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$. If considering the negative feedback gain $A_V$ of the amplifier as shown in Fig. 15.6, then the transfer function $T(s)$ shall be $T(s) = \left(1 + \frac{R_1}{R_2}\right) \frac{1}{1 + sRC}$ and its magnitude shall be $|T(s)| = \left(1 + \frac{R_1}{R_2}\right) \frac{1}{\sqrt{1 + (\omega RC)^2}}$. The phase is $\phi = -\tan^{-1}(\omega RC)$.

![Figure 15.6: Single-pole low-pass active filter](image)

The magnitude of the transfer function $|T(s)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$ can be re-written as $|T(s)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^2}}$. If the internal critical frequency of the operational amplifier is much larger than $f_c$ of the low-pass filter, then the transfer function $T(s)$ which is also the voltage gain $A_V$ will roll-off at the rate of -20dB/decade/pole which is shown in Fig. 15.7. Its Bode plot function shall be
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\[ |T(s)|_{dB} = 20 \log_{10} \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^2}} \quad (15.11) \]

Equation (15.6) can be re-written as

\[ 20 \log_{10}(1) - 20 \log_{10}\sqrt{1 + \left(\frac{f}{f_c}\right)^2} = 20 \log_{10}(1) - 10 \log_{10}\left[1 + \left(\frac{f}{f_c}\right)^2\right]. \]

**Gain T(s) (dB)**

![Figure 15.7: The voltage gain response of low-pass active filter](image)

As it has been mentioned earlier, Butterworth filter exhibits very flat amplitude in its pass band. For this reason it is also called *maximally flat filter*. Butterworth filter utilizing two RC networks, which is also called *two-pole* or *second order filter*, is shown in Fig. 15.8. Its roll-off rate is -40dB/decade. Substituting \( Y_1 = 1/R_1 \), \( Y_2 = 1/R_2 \), \( Y_3 = sC_1 \), and \( Y_4 = sC_2 \) into the general transfer function \( T(s) \) shown in equation (15.1), the transfer function of the filter is

\[ T(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1/R_1 \cdot 1/R_2}{1/R_1 \cdot 1/R_2 + sC_2/(1/R_1 + 1/R_2 + sC_1)}. \]

This transfer function can be re-arranged as

\[ T(s) = \frac{R_1R_2C_1C_2}{s^2 + \frac{C_2}{s}(R_1 + R_2) + \frac{1}{R_1R_2C_1C_2}}. \]

The transfer function can further be arranged as the standard second-order low-pass network equation, which is
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\[ T(s) = \frac{\omega_c^2}{s^2 + 2\xi\omega_c s + \omega_c^2} \]  (15.12)

where \( \omega_c = \frac{1}{R_1 R_2 C_1 C_2} \) is the critical angular frequency \( \omega_c \) and the damping factor \( \xi_{LP} \) is equal to

\[ \xi_{LP} = \frac{R_2 C_2 + R_1 C_2}{2\sqrt{R_1 R_2 C_1 C_2}} \]  (15.13)

If \( R_1 = R_2 = R, \tau_1 = RC_1, \tau_2 = RC_2 \), then the transfer function \( T(s) \) shall be

\[ T(s) = \frac{1}{1 + sC_2 \left( 2R + sC_1 R^2 \right)} = \frac{1}{\left( 1 - \omega^2 \tau_1 \tau_2 \right) j \left( 2\omega \tau_2 \right)} \]  (15.14)

The magnitude of the transfer function shall be

\[ |T(s)| = \frac{1}{\sqrt{\left( 1 - \omega^2 \tau_1 \tau_2 \right)^2 + 4\omega^2 \tau_2^2}} \]  (15.15)

The critical frequency \( f_c \) of the filter shall be

\[ f_c = \frac{1}{2\pi\sqrt{\tau_1 \tau_2}} \]  (15.16)

In order to maintain minimum rate of change, which is a flat filter for the Butterworth filter, the derivation of \( T(s) \) with respect to \( \omega \) for \( \omega = 0 \), which has dc only, should be equal to zero i.e. \( \left. \frac{dT(s)}{d\omega} \right|_{\omega=0} = 0 \). From equation (15.15), \( \left. \frac{dT(s)}{d\omega} \right|_{\omega=0} \) equals to

\[ \left. \frac{dT(s)}{d\omega} \right| = -\frac{1}{2} \left[ 1 - \omega^2 \tau_1 \tau_2 \right]^2 \left[ 2(1 - \omega^2 \tau_1 \tau_2)2\omega \tau_1 \tau_2 + 8\omega^2 \tau_2 \right] \]  (15.17)

then

\[ \left. \frac{dT(s)}{d\omega} \right| = \left[ -4(1 - \omega^2 \tau_1 \tau_2)\omega \tau_1 \tau_2 + 8\omega^2 \tau_2 \right] = 0 \]  (15.18)
When $\omega = 0$, equation (15.18) yields $C_1 = 2C_2$. This shall mean that equation (15.15) becomes

$$|T(s)| = \frac{1}{\sqrt{1 + 4(\omega \tau)^2}}$$

(15.19)

Substituting $C_1 = 2C_2$ in to equation (15.16), it yields critical frequency $f_c = \frac{1}{2\pi \sqrt{2}\tau} = \frac{1}{2\pi \sqrt{2R}\tau}$. Likewise the critical frequency is also equal to $f_c = \frac{1}{\pi \sqrt{2R}\tau}$ after substituting $C_1 = 2C_2$. Substituting $f_c = \frac{1}{2\pi \sqrt{2}\tau}$ into equation (15.19), its yields the magnitude of the transfer function as $|T(s)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^4}}$.

Substituting $C_1 = 2C_2$ and $R_1 = R_2 = R$ into equation (15.13), which is

$$\xi_{LP} = \frac{R_2C_2 + R_1C_1}{2\sqrt{R_1R_2C_1C_2}}$$

it yields damping factor $\xi_{LP}$ of $\frac{1}{\sqrt{2}} = 0.707$ for Butterworth low-pass active filter.

Let’s now consider a Sallen-Key equal component low-pass active filter shown in Fig. 15.9. Equal component shall mean the value of $R_1 = R_2 = R$ and $C_1 = C_2 = C$. Based on the general two-pole equation (15.9), which is $T(s) =$
\[
\frac{V_{out}(s)}{V_{in}(s)} = \frac{A_VZ_1Z_4}{Z_1Z_2 + Z_2Z_3 + Z_3Z_4 + Z_4Z_1 + Z_1Z_4(1-A_V)},
\]

after substituting R and C, the transfer function \(T(s)\) becomes

\[
\frac{V_{out}(s)}{V_{in}(s)} = \frac{A_V1/s^2C^2}{R^2 + R/sC + 1/s^2C^2 + R/sC + R/sC(1-A_V)}
\]

\[
= \frac{A_V / s^2C^2}{s^2 + s(3-A_V)/CR + 1/C^2R^2}.
\]

As the standard second-order low-pass network equation, the transfer function shall be

\[
T(s) = \frac{A_V \omega_c^2}{s^2 + 2\xi \omega_c s + \omega_c^2},
\]

where the critical angular frequency \(\omega_c = \frac{1}{RC}\) and \(2\xi \omega_c = (3-A_V)/CR\). This will give rise to damping factor \(\xi_{LP} = 0.5(3-A_V)\). This shall mean that the pass-band gain \(A_V\) is equal to \(A_V = 3 - 2\xi_{LP}\). Also values of \(R_3\) and \(R_4\) are determined to be \(R_3 = 2RA_V\) and \(R_4 = \frac{R_3}{A_V - 1}\). \(R_3\) is determined from \(R_3||R_4 = R_1 + R_2 = 2R\) that used for offset current error.

A third order Butterworth low-pass filter is shown in Fig. 15.10 respectively. For \(R_1 = R_2 = R_3\), the critical frequency of the third order Butterworth low-pass filter is

\[
f_c = \frac{1}{2\pi \sqrt{\tau_1\tau_2\tau_3}} \quad (15.20)
\]
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Its magnitude shall be \(|T(s)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^6}}.\)

Figure 15.10: Third order Butterworth low-pass filter of \(f_c = 1.0\text{kHz}\)

The capacitance can be scaled inversely to get the other critical frequency. The circuit shown in Fig. 15.10 has critical frequency of 1.0 kHz. The critical frequency shall be 4.0kHz if all the values of capacitors \(C_1, C_2,\) and \(C_3\) are decreased by four order and maintaining the value of resistor. i.e. \(C_1 = 0.02\mu\text{F}/4, C_2 = 0.005\mu\text{F}/4,\) and \(C_3 = 0.01\mu\text{F}/4.\) This claim can be verified using equation (15.20).

15.4 High-Pass Filter

The basic passive high-pass filter is shown in Fig. 15.11. As compared with the low-pass filter circuit shown in Fig. 15.4, the basic difference is the exchange position of capacitor and resistance.

Figure 15.11: Passive high-pass filter circuit
The output response for different frequency is shown in Fig. 15.12. For input frequency $f$ much smaller than critical frequency $f_c$, the output is approximately zero volt.

$$V_{in} \rightarrow C \rightarrow R \rightarrow V_{out}$$

At $f < f_c$, $V_{out} \approx 0$ V

At $f < f_c$, $V_{out} < V_{in}$

$$V_{in} \rightarrow C \rightarrow R \rightarrow V_{out}$$

At $f = f_c$, $V_{out} = 0.707 \times V_{in}$

At $f > f_c$, $V_{out} \approx V_{in}$

**Figure 15.12:** The expected output signal of the basic high-pass filter with input of various frequency

The basic active high-pass circuit and the output response with respect to frequency are shown in Fig. 15.13.

**Figure 15.13:** (a) A basic high-pass filter and (b) its output-frequency response
The transfer function $T(s)$ of the filter shall be

$$T(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{sRC}{1 + sRC} = \frac{1}{1 + \frac{1}{sRC}}$$

(15.21)

The magnitude of the transfer function $T(s)$ shall be

$$|T(s)| = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^2}}$$

(15.22)

The phase is $\phi = \tan^{-1}\left(\frac{\omega RC}{0}\right)$ - $\tan^{-1}(\omega RC) = 90^0 - \tan^{-1}(\omega RC)$. The critical frequency $f_c$ shall be equal to $\frac{1}{2\pi RC}$.

Substituting $Y_1 = sC_1$, $Y_2 = sC_2$, $Y_3 = 1/R_1$, and $Y_4 = 1/R_2$ into the general transfer function $T(s)$ shown in equation (15.1), the transfer function of for two-pole Butterworth active high-pass filter is $T(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{sC_1 \cdot sC_2}{sC_1 \cdot sC_2 + 1/R_2 (sC_1 + sC_2 + 1/R_1)}$. This transfer function can be re-arranged as

$$T(s) = \frac{s^2}{s^2 + \frac{R_1 (C_1 + C_2)}{R_1 R_2 C_1 C_2} s + \frac{1}{R_1 R_2 C_1 C_2}}$$

and further arrange as

$$T(s) = \frac{s^2}{s^2 + 2\xi \omega_c s + \omega_c^2}$$

(15.23)

where $\omega_c = \frac{1}{R_1 R_2 C_1 C_2}$ is the critical angular frequency and the damping factor $\xi_{\text{HP}}$ is equal to

$$\xi_{\text{HP}} = \frac{R_1 C_1 + R_2 C_2}{2 \sqrt{R_1 R_2 C_1 C_2}}$$

(15.24)
If \(C_1 = C_2 = C\), then the transfer function \(T(s)\) shall be equal to \(T(s) = \frac{1}{1 - \frac{1}{\omega^2 R_1 R_2 C^2} + \frac{2}{j\omega R_2 C}}\).

From equation \(T(s) = \frac{1}{1 - \frac{1}{\omega^2 R_1 R_2 C^2} + \frac{2}{j\omega R_2 C}}\), the critical frequency \(f_c\) is

\[
f_c = \frac{1}{2\pi \sqrt{\tau_1 \tau_2}}
\]

where \(\tau_1 = R_1 C, \tau_2 = R_2 C\). The magnitude of the transfer function \(T(s)\) shall be equal to

\[
|T(s)| = \frac{1}{\sqrt{\left(1 - \frac{1}{\omega^2 \tau_1 \tau_2}\right)^2 + \left(\frac{2}{\omega \tau_2}\right)^2}}
\]

In order to maintain minimum rate of change, which is a flat filter for the Butterworth filter, the derivation of \(T(s)\) with respect to \(\omega\) for \(\omega = 0\) (dc level only) should be equal to zero i.e. \(\frac{dT(s)}{d\omega} = 0\). From equation (15.26), \(\frac{dT(s)}{d\omega}\) equals to

\[
\frac{dT(s)}{d\omega} = -\frac{1}{2} \left[\left(1 - \frac{1}{\omega^2 \tau_1 \tau_2}\right)^2 + \left(\frac{2}{\omega \tau_2}\right)^2\right]^{3/2} \left[2 \left(1 - \frac{1}{\omega^2 \tau_1 \tau_2}\right) \left(\frac{2}{\omega \tau_2}\right) - \frac{8}{\omega^3 \tau_2^2}\right]
\]

then

\[
\frac{dT(s)}{d\omega} = \left[2 \left(1 - \frac{1}{\omega^2 \tau_1 \tau_2}\right) \left(\frac{2}{\omega^3 \tau_2}\right) - \frac{8}{\omega^3 \tau_2^2}\right]
\]

For \(\omega = 0\), equation (15.28) yields \(2R_1 = R_2\). This shall mean that magnitude equation (15.26) becomes
Substituting $2R_1 = R_2$ into equation (15.28) yields critical frequency $f_c = \frac{1}{2\pi\sqrt{2}\tau_1} = \frac{1}{2\pi\sqrt{2}R_1C}$. Likewise the critical frequency is also equal to $f_c = \frac{1}{\pi\sqrt{2}R_2C}$ after substituting $2R_1 = R_2$. Substituting $f_c = \frac{1}{2\pi\sqrt{2}\tau_1}$ into equation (15.29), its yields the magnitude of the transfer function as $|T(s)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^4}}$. Substituting $C_1 = C_2$ and $2R_1 = R_2 = R$ into equation (15.24), which is $\xi_{HP} = \frac{R_1C_1 + R_1C_2}{2\sqrt{R_1R_2C_1C_2}}$, it yields damping factor $\xi_{HP}$ of $\frac{1}{\sqrt{2}} = 0.707$ for Butterworth high-pass active filter.

Fig. 15.14 shows the two-pole Butterworth high-pass filter where it has $R_2$ value equals to twice the value of resistor $R_1$ and same capacitor values for $C_1$ and $C_2$.

Let’s now consider a Sallen-Key equal component high-pass active filter shown in Fig. 15.15. Equal component shall mean the value of $R_1 = R_2 = R$ and $C_1 = C_2 = C$. Based on the general two-pole equation (15.9), which is $T(s) =$
\[
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{A_v Z_3 Z_4}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_4 + Z_4 Z_1 + Z_1 Z_2 (1 - A_v)},
\]

after substituting R and C, the transfer function \( T(s) \) becomes

\[
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{A_v R^2}{1/s^2 C^2 + R/R C + R^2 + R/R C + R/s C + R/s C (1 - A_v)}
\]

\[
= \frac{A_v s^2}{s^2 + s(3 - A_v)/CR + 1/C^2 R^2}.
\]

As the standard second-order low-pass network equation, the transfer function shall be

\[
T(s) = \frac{A_v s^2}{s^2 + 2\xi \omega_c s + \omega_c^2},
\]

where the critical angular frequency \( \omega_c = \frac{1}{RC} \) and \( 2\xi \omega_c = (3 - A_v)/CR \). This will give rise to damping factor \( \xi_{\text{HP}} = 0.5(3 - A_v) \). This shall mean that the pass-band gain \( A_V \) is equal to

\[
A_V = 3 - 2\xi_{\text{HP}}.
\]

Also values of \( R_3 \) and \( R_4 \) are

\[
R_3 = R_A V \quad \text{and} \quad R_4 = \frac{R_3}{A_V - 1}.
\]

\( R_3 \) is determined from \( R_3||R_4 = R_2 = R \) that used for offset bias current.

**Figure 15.15:** Sallen-Key equal component high-pass active filter

A three-pole Butterworth high-pass filter with \( C_1 = C_2 = C_3 \), the critical frequency \( f_c \) shall be

\[
\frac{1}{2\pi \sqrt[4]{R_1 R_2 R_3 C_1 C_2 C_3}}
\]

(15.30)
and its filter circuit is shown in Fig. 15.16. Its magnitude shall be 
$$|T(s)| = \frac{1}{\sqrt{1 + \left(\frac{f_c}{f}\right)^6}}.$$ 
Thus, a higher order filter, a general equation for its magnitude shall be 
$$|T(s)| = \frac{1}{\sqrt{1 + \left(\frac{f_c}{f}\right)^{2N}}}$$ 
where N indicates the number of pole.

![Filter Circuit Diagram](image)

**Figure 15.16:** Three-pole Butterworth high-pass filter of $f_c = 1.0$kHz

Note that the resistance can be scaled inversely to get the other critical frequency. For an example from the circuit shown in Fig. 15.16, the critical frequency shall be 500Hz if all the values of resistors $R_1$, $R_2$, and $R_3$ are double and maintaining the value of capacitor.

### 15.5 Higher Order Active Filter

The higher order active filter such as three-order has been discussed in terms of its transfer function and the roll-off value of the pass-band gain $A_v$. The discussion of the design approach has yet to be formulated. Higher order filter is normally designed using cascaded approach. For example, a three-order active filter is designed by cascading a two-order filter with a one-order filter. A fifth order filter is designed by cascading two two-order filters and a one order filter. When cascading the filter, the damping factor $\xi$ and frequency correction factor $\kappa$ are usually different from those given for a single-stage second-order filter.

The damping factor $\xi$ and low-pass frequency correction factors $\kappa_{LP}$ for higher order low-pass active filter is shown in Fig. 15.17.
<table>
<thead>
<tr>
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<th>Section Order</th>
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**Figure 15.17:** The damping factor $\xi$ and low-pass frequency correction factors $\kappa_{LP}$ of higher order low-pass active filter.

The frequency correction factor for a high-pass filter $\kappa_{HP}$ is the reciprocal of the low-pass frequency correction factor $\kappa_{LP}$. i.e.

$$\kappa_{HP} = 1/\kappa_{LP}$$ (15.31)

The corrected frequency $f_o$ is equal to the ratio of cut-off frequency $f_c$ and the frequency correction factor $\kappa$ as shown in equation (15.32).

$$f_o = f_c/\kappa$$ (15.32)
**Example 15.2**
Design a fourth-order low-pass Sallen-Key Chebyshev active filter with critical frequency at 10kHz.

**Solution**
This fourth-order low-pass filter can be achieved by cascading two second-order Sallen-Key low-pass active filter.

The parameter for first stage shall be $\xi = 0.6375$ and $\kappa_{LP} = 1.992$.
Thus, the pass-band gain of first stage shall be $A_{V1} = 3 - 2\xi_{LP} = 1.725V/V$.
The corrected cut-off frequency is $f_o = 10kHz/1.992 = 5.02kHz$.

Let $C = 0.01 \mu F$, then $R = \frac{1}{2\pi f_o C} = 3.17k\Omega$.
$R_{11} = 2RA_{V1} = 2x3.17k\Omega(1.725) = 10.9k\Omega$
$R_{12} = \frac{R_{11}}{A_{V1} - 1} = \frac{10.9k\Omega}{1.725 - 1} = 15.0k\Omega$.

The parameter for second stage shall be $\xi = 0.1405$ and $\kappa_{LP} = 1.06$.
Thus, the pass-band gain of first stage shall be $A_{V2} = 3 - 2\xi_{LP} = 2.719 V/V$.
The corrected cut-off frequency is $f_o = 10 kHz/1.06 = 9.43 kHz$.

Let $C = 0.01 \mu F$, then $R = \frac{1}{2\pi f_o C} = 1.68k\Omega$.
$R_{21} = 2RA_{V2} = 2x1.68 k\Omega(2.719) = 9.1k\Omega$.
$R_{22} = \frac{R_{21}}{A_{V2} - 1} = \frac{9.1k\Omega}{2.719 - 1} = 5.3k\Omega$.

The overall pass-band gain is $A_{V1}xA_{V2} = 1.725x2.719 = 4.69V/V$.
dBG = $20 \log(4.69) = 13.4dB$.

**15.6 Band-Pass Filter**

A band-pass filter passes all signals lying within a band between a low and high critical frequency limits and rejects all other frequencies outside this band. Figure 15.18 illustrates band-pass filter response curve. The overall pass-band gain $A_V$ is the product of the two individual low-pass and high-pass pass-band gain $A_{V1}$ and $A_{V2}$.
The frequency in the center of the pass band is called the *center frequency* or *resonant frequency* $f_r$, which is defined the geometric average of the lower and upper critical frequencies. Thus,

$$f_r = \sqrt{f_{c1} f_{c2}}$$  \hspace{1cm} (15.33)

The quality factor $Q$ of a band-pass filter is the ratio of the center frequency to the bandwidth. It is also equal to the reciprocal of two-time the damping factor $\xi$.

$$Q = \frac{f_r}{BW} = \frac{1}{2\xi}$$  \hspace{1cm} (15.34)

The $Q$ value is a selectivity of a band-pass filter. If $Q$ value is small, it shall mean the more frequencies will be passed. Likewise, if the $Q$ value is large then bandwidth is narrow. Generally speaking $Q$ value $< 10$ means wide-band filter. $Q$ value $\geq 10$ means narrow-band filter.

Band-pass filter can be achieved by combining a high-pass and low-pass filter. A 2-pole high-pass and low-pass filter cascaded for a band-pass filter is shown in Fig. 15.19.
Figure 15.19: A band-pass filter forms by cascading a 2-pole high-pass and a 2-pole low-pass filter

Band-pass filter also can be achieved by multiple-feedback. Figure 15.20 shows a multiple-feedback band-pass filter whereby $C_1$ and $R_1$ provide low-pass filter and $C_2$ and $R_2$ provide high-pass filter.

Figure 15.20: A multiple-feedback band-pass filter

From the circuit $i_3 = -\frac{V_{out}}{R_3}$; $V_a = V_{C2} = \frac{i_3}{sC_2} = -\frac{V_{out}}{sR_2C_2}$; $i_2 = V_a/R_2 = -\frac{V_{out}}{sR_2R_3C_2}$. 

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$$i_a = \frac{(V_a - V_{out})}{1/sC_1} = (V_a - V_{out})sC_1 = \left( -\frac{V_{out}}{sR_3C_2} \right) sC_1 = -\frac{V_{out}}{R_3} \left( \frac{C_1}{C_2} + sR_3C_1 \right);$$

$$i_1 = \frac{V_{in} - V_a}{R_1} = \frac{V_{in}}{R_1} + \frac{V_{out}}{sR_3C_2}.$$ 

From KCL, current at node a follows equation $i_1 = i_2 + i_3 + i_4$. Thus, it is

$$\frac{V_{in}}{R_1} + \frac{V_{out}}{sR_3C_2} = \frac{-V_{out}}{sR_2R_3C_2} - \frac{V_{out}}{R_3} \left( \frac{C_1}{C_2} + sR_3C_1 \right).$$

After re-arranging the equation, the transfer function $T(s) = \frac{V_{out}}{V_{in}} = -\frac{sC_2R_3C_1}{(R_1R_2R_3C_1C_2)s^2 + s(R_1R_2C_2 + R_1R_2C_1) + (R_1 + R_2)}$. Dividing the transfer function $T(s)$ by the factor $(R_1R_2R_3C_1C_2)$, it becomes

$$T(s) = -\frac{s/R_1C_1}{s^2 + \frac{(C_1 + C_2)}{R_3C_1C_2}s + \frac{R_1 + R_2}{R_1R_2R_3C_1C_2}}.$$  

Equation (15.35) follows the standard band-pass network transfer function $T(s) = A_r(2\xi\omega_0s)\frac{s}{s^2 + 2\xi\omega_0s + \omega^2_0}$, where $\omega_0$ is resonant angular frequency and $A_r$ is the gain at resonant frequency.

Based on equation (15.35), the resonant frequency $f_r$ is

$$f_r = \frac{1}{2\pi \sqrt{(R_1 || R_2)R_3C_1C_2}}.$$  

If capacitance $C_1 = C_2 = C$ then

$$f_r = \frac{1}{2\pi C \left[ \frac{R_1 + R_2}{R_1R_2R_3} \right]^{1/2}}.$$  

From the transfer $T(s)$ and the equation of standard band-pass network, $A_r(2\xi\omega_0) = 1/R_1C_1$ and $\frac{C_1 + C_2}{R_3C_1C_2} = 2\xi\omega_0$. If the capacitance value is chosen to be $C$ then the value of resistance $R_1$, $R_2$, and $R_3$ shall follow the equation (15.38), (15.39), and (15.40).
The resonant angular frequency is
\[ \omega_r = \sqrt{\frac{R_1 + R_2}{R_1 R_2 C^2}}. \]
Substituting equation (15.38) and (15.39) into this equation yields,
\[ R_2 = \frac{Q}{\omega_r C (2Q^2 - A_r^2)} \]  
(15.40)

From equation (15.38) and (15.39), the resonant gain \( A_r \) is
\[ A_r = \frac{R_3}{2R_1} \]  
(15.41)

From equation (15.40), notice that the condition that \( A_r < 2Q^2 \) must be satisfied. This is the condition limit for the resonant gain \( A_r \).

From equation (15.39), \( R_3 \) is \( R_3 = \frac{2Q}{\omega_r C} \). This shall mean that the resonant angular frequency \( \omega_r \) is \( \omega_r = \frac{2Q}{R_3 C} \). Knowing that bandwidth \( BW \) and quality factor relation is \( Q = \frac{\omega_r}{BW} \), the bandwidth \( BW \) is
\[ BW = \frac{2}{R_3 C} \]  
(15.42)

### 15.7 Band-Stop Filter

Band-stop filter is also known as notch, band-reject or band-elimination filter. Its operation is opposite manner of band-pass filter. It rejects frequencies lying in a band and passes frequencies other the band. The illustration of the filter's response is shown in Fig. 15.21. It has two critical frequencies \( f_{c1} \) and \( f_{c2} \), where \( f_{c1} \) is critical frequency for low-pass and \( f_{c2} \) is the critical frequency for high-pass.
The frequency in the center of the pass band is called the *center frequency* or *resonant frequency* $f_r$, which is defined the geometric average of the lower and upper critical frequencies i.e. $f_r = \sqrt{f_{c1}f_{c2}}$.

**Figure 15.21:** Frequency response of band-stop filter

A multiple-feedback band-stop filter is shown in Fig. 15.22. This approach creates loading effect because additional RC circuits. The better approach is using multiple cascaded operational amplifiers because the output impedance of the operational amplifier is very low that has no loading effect. This approach is equally applicable for other higher order active filter circuits.

**Figure 15.22:** A multiple-feedback band-stop filter
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The approach of deriving the key parameters of the filter is same as the band-pass filter.

The resonant frequency of the filter is

\[ f_r = \frac{1}{2\pi \sqrt{R_1 R_2 C_1 C_2}} \]  \hspace{1cm} (15.43)

If \( C_1 = C_2 = C \) then

\[ f_r = \frac{1}{2\pi C \sqrt{R_1 R_2}} \]  \hspace{1cm} (15.44)

The bandwidth \( BW \) shall be determined from expression

\[ BW = \frac{1}{\pi R_1 C} \]  \hspace{1cm} (15.45)

15.8 All-Pass Filter

An all-pass filter passes all frequency components of the input signals without attenuation. However, it provides predictable phase shift for different frequencies of the input signals. All-pass filter is commonly used to compensate for phase change and is also called as delay equalizer or phase corrector. One of the applications is correct the phase shift of telephone transmission line. The circuit of the all-pass filter is shown in Fig. 15.23.

**Figure 15.23:** An all-pass filter
Using the superposition method, the complex output voltage $V_{\text{out}}(s)$ is equal to

$$V_{\text{out}}(s) = -\frac{R_1}{R_2}V_{\text{in}}(s) + \frac{1/sC}{R + 1/sC} \left( 1 + \frac{R_1}{R_2} \right) V_{\text{in}}(s) \quad (15.46)$$

Thus, the transfer function $T(s)$ shall be

$$T(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \left[ \frac{1}{sCR + 1 \left( 1 + \frac{R_1}{R_2} \right) - \frac{R_1}{R_2}} \right] \quad (15.47)$$

If $R_1 = R_2 = R$, then the transfer function is $T(s) = \frac{1-sCR}{1+sCR}$. The transfer function can also be written as $T(j\omega) = \frac{1-j\omega CR e^{-j\theta_1}}{1+j\omega CR e^{j\theta_2}} = \frac{\sqrt{1+\omega^2 C^2 R^2 e^{-j\theta_1}}}{\sqrt{1+\omega^2 C^2 R^2 e^{j\theta_2}}} = \exp(-2\theta)$ for $\theta_1 = \theta_2 = \theta$. Thus, the phase $\theta$ for the filter is

$$\theta = -2\tan^{-1}(\omega CR) \quad (15.48)$$

This shall also mean that the output is lagged behind the input.

**15.9 Filter Design Guidelines**

Designing filters requires selecting the values of $R$ and $C$ that will satisfy the cut-off frequency $f_c$, bandwidth $BW$, and pass-band gain $A_v$. The guidelines provided here will help in the design.

Step 1 decides the specification, which includes response type i.e. Butterworth, Bessel, or Chebyshev, cutoff frequency $f_c$, bandwidth $BW$, damping factor $\xi$, frequency correction factor $\kappa$, and order of the filter, whereby these data can be obtained from Fig. 15.17, or derived equation (15.31) and (15.32).

Step 2 decides the capacitor value, which is normally in between 0.01$\mu$F to 0.1 $\mu$F. Mylar or tantalum types are preferred for performance.

Step 3 finds resistor value in the practical range in between 1.0k$\Omega$ to 500k$\Omega$. If the range does not meet the design requirement, then change the value of capacitor. When choosing the value for resistor, offset current
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correction is necessary. You may refer to Chapter 9 or example 15.2 to help you.

Step 4: If it is necessary to change the cutoff frequency, *frequency scaling* can be applied. This can be done by multiplying the value of R or C but not both at the same time by the ratio of original cutoff frequency \( f_c \) to new cutoff frequency \( f_n \). The new value of R or C can be found using equation

\[
R_n \text{ (or } C_n \text{)} = \frac{\text{Original cutoff frequency } f_c \times R \text{ (or } C \text{)}}{\text{New cutoff frequency } f_n}
\]  

(15.49)

**Exercises**

15.1. Show that circuit below is a low-pass filter. Calculate the corner frequency \( f_c \) if \( L = 2\text{mH}, C = 10\mu F, \text{ and } R = 10\text{k}\Omega \).

![Low-pass filter circuit diagram]

15.2. Derive a complex transfer function \( T(s) \) for a 1-pole high-pass filter element and the equation for its magnitude. Draw a Bode plot for the magnitude of the transfer function for frequency range from \( 0.1f_c \) to \( 10f_c \).

15.3. Derive a complex transfer function \( T(s) \) for a 1-pole low-pass filter element and the equation for its magnitude. Draw a Bode plot for the magnitude of the transfer function for frequency range from \( 0.1f_c \) to \( 10f_c \).

15.4. Draw the phase shift diagrams for Q15.2 and Q15.3.

15.5. Design a two-pole low-pass Butterworth for an audio amplifier application such that it has bandwidth 20kHz.

15.6. A wideband band-pass filter circuit is shown in the Figure. Show that the transfer function is

\[ T(S) = \frac{R_2}{R_1} \frac{j\omega R_1 C_1}{(1 + j\omega R_1 C_1)(1 + j\omega R_2 C_2)}. \]
15.7. Using the circuit shown as the guide, design a first-order low-pass active filter that has cut-off frequency of 5.0kHz with a pass-band gain of 16dB.

15.8. Determine the pass-band gain $A_V$, cut-off frequency $f_c$, damping factor $\xi$, and response type for the third-order high-pass filter shown in the figure.
15.9. Design a band-pass active filter with cut-off frequency $f_c = 5.0\text{kHz}$, resonant gain $A_r = 5$, and quality factor $Q = 5$.

Bibliography